CHAPTER FIVE

CONTINUOUS SYSTEMS. EXACT SOLUTIONS

5.1 GENERAL DISCUSSION

Chapter 4 was devoted exclusively to the vibration of discrete systems, whereas this chapter is devoted to continuous systems. This fact should not be interpreted, however, as an indication that discrete and continuous systems represent different types of systems exhibiting dissimilar dynamical characteristics. In reality the opposite is true, as discrete and continuous systems represent merely two mathematical models of identical physical systems. The basic difference between discrete and continuous systems is that discrete systems have a finite number of degrees of freedom and continuous systems have an infinite number of degrees of freedom. This results from the fact that the index i identifying a typical lumped mass has as counterpart an independent spatial variable x identifying the nominal position of an infinitesimal mass element. Consistent with this, discrete systems are governed by ordinary differential equations and continuous systems by partial differential equations. Nevertheless, because discrete and continuous systems represent in general models of identical physical systems, they display similar dynamical behavior.

This chapter begins by stressing the intimate relation between discrete and continuous systems. In fact, the mathematical formulation for a given continuous system is derived as a limiting case of that of a discrete system. The discussion continues by showing that various concepts introduced in our study of discrete systems have their counterparts in continuous systems. Indeed, to a finite set of eigenvalues and finite-dimensional eigenvectors corresponds an infinite set of eigenvalues and space-dependent eigenfunctions. Concepts such as the orthogonality of natural modes of vibration and the ensuing expansion theorem can be defined for continuous systems in a manner analogous to that for discrete systems.

In this chapter, a number of continuous systems are discussed, such as strings in transverse vibration, rods in axial vibration, shafts in torsion, and bars in bending. Strings, rods, and shafts are governed by second-order differential equations in space and are analogous in nature. On the other hand, bars are governed by fourth-order differential equations. Exact solutions for the vibration of continuous systems can be obtained only in special cases, mainly when the system parameters are uniformly distributed. In this case, second-order differential equations in space reduce to the so-called "wave equation."

5.2 RELATION BETWEEN DISCRETE AND CONTINUOUS SYSTEMS. BOUNDARY-VALUE PROBLEM

As pointed out in **Sec.5.1**, there is a very intimate relation between discrete and continuous systems, as they generally represent two distinct mathematical models of the same physical system. To demonstrate this, we derive the differential equation for the transverse vibration of a string first by regarding it as a discrete system and letting it approach a continuous model in the limit. Then, we formulate the problem by regarding the system as continuous from the beginning.

Let us consider a system of discrete masses m_i $(i = 1, 2, \cdots, n)$ connected by massless strings, where the masses m_i are subjected to the external forces F_i, as shown in **Fig.5.la**. To derive the differential equation of motion for a typical mass m_i, we concentrate our attention on the three adjacent masses m_{i-1}, m_i and m_{i+1} of **Fig.5.1b**. The tensions in the string segments connecting m_i to m_{i-1} and m_{i+1} are denoted by T_{i-1} and T_i, and the horizontal projections of these segments by Δx_{i-1} and Δx_i, respectively. The displacements $y_i(t)$ $(i = 1, 2, \cdots, n)$ of the masses m_i are assumed to be small, so that the projections Δx_i remain essentially unchanged during motion. Moreover, the angles between the string segments and the horizontal are sufficiently small that the sine and tangent of the angles are approximately equal to one another. Hence, using Newton's second law, the equation of motion of the mass m_i in the vertical direction has the form

$$T_i \frac{y_{i+1}-y_i}{\Delta x_i} - T_{i-1}\frac{y_i - y_{i-1}}{\Delta x_{i-1}} + F_i = m_i \frac{d^2 y_i}{dt^2} \tag{5.1}$$

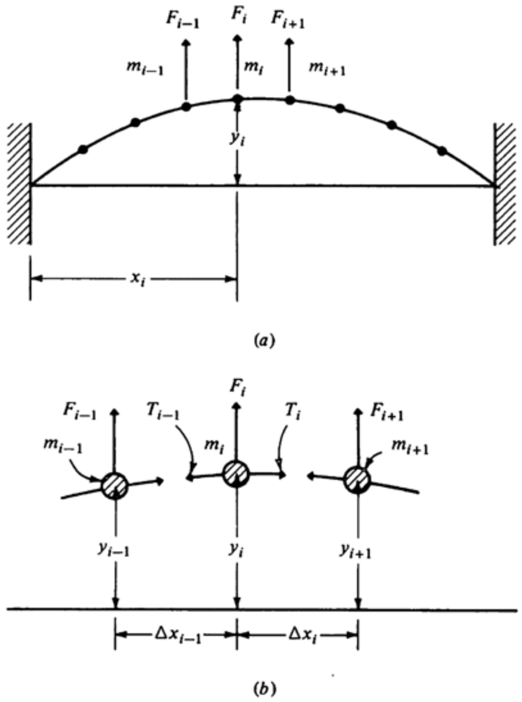


Figure 5.1

Equation (5.1) is applicable to any mass m_i $(i = 2, 3, \cdots, n-1)$. The equation can also be used for $i=1$ and $i=n$, but certain provisions must be made to reflect the way the system is supported, as we shall see shortly. Rearranging Eq.(5.1), we obtain the set of simultaneous ordinary differential equations

$$\frac{T_i}{\Delta x_i} y_{i+1} - \left(\frac{T_i}{\Delta x_i} + \frac{T_{i-1}}{\Delta x_{i-1}}\right) y_i + \frac{T_{i-1}}{\Delta x_{i-1}} y_{i-1} + F_i = m_i \frac{d^2 y_i}{dt^2} \quad (i = 1, 2, \cdots, n) \tag{5.2}$$

in the variables $y_i(t)$ $(i = 1, 2, \cdots, n)$, and we notice that the equations for $i=1$ and $i=n$ contain the displacements y_0 and y_{n+1}, respectively. If the string is fixed at both ends, as is the case with the system shown in **Fig.5.la**, then we must set

$$y_0(t) = y_{n+1}(t) = 0 \tag{5.3}$$

in Eqs.(5.2). In other cases different conditions are possible. Indeed, if the ends $x=0$ and $x=L$ are attached to vertical springs, or if they are free to move along a vertical line, the end conditions must reflect the fact that there is a force proportional to the stretching of the spring, or that the vertical component of the force at that particular end is zero. We shall not pursue this subject any further at this time but return to Eq.(5.1), because our object is to draw the analogy between discrete and continuous systems.

If we introduce the notation $y_{i+1} - y_i = \Delta y_i$, $y_i - y_{i-1} = \Delta y_{i-1}$, Eq.(5.1) becomes

$$T_i \frac{\Delta y_i}{\Delta x_i} - T_{i-1}\frac{\Delta y_{i-1}}{\Delta x_{i-1}} + F_i = m_i \frac{d^2 y_i}{dt^2} \quad (i = 1, 2, \cdots, n) \tag{5.4}$$

But the first two terms on the left side of Eq.(5.4) constitute the incremental change in the vertical force component between the left and right sides of m_i. In view of this, we can write Eq.(5.4) as

$$\Delta\left(T_i\frac{\Delta y_i}{\Delta x_i}\right)+F_i=m_i\frac{d^2y_i}{dt^2}\qquad(i=1,2,\cdots,n)\tag{5.5}$$

Moreover, dividing both sides of (5.5) by Δx_i, we arrive at

$$\frac{\Delta}{\Delta x_i}\left(T_i\frac{\Delta y_i}{\Delta x_i}\right)+\frac{F_i}{\Delta x_i}=\frac{m_i}{\Delta x_i}\frac{d^2y_i}{dt^2}\qquad(i=1,2,\cdots,n)\tag{5.6}$$

At this time we let the number n of masses m_i increase indefinitely, while the masses themselves and the distance between them decrease correspondingly, and replace the indexed position x_i by the independent spatial variable x, so that in the limit, as $\Delta x_i\to 0$, Eq.(5.6) reduces to

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right]+f(x,t)=\rho(x)\frac{\partial^2y(x,t)}{\partial t^2}\tag{5.7}$$

which must be satisfied over the domain $0<x<L$, where

$$f(x,t)=\lim_{\Delta x_i\to 0}\frac{F_i}{\Delta x_i},\quad \rho(x,t)=\lim_{\Delta x_i\to 0}\frac{m_i}{\Delta x_i}\tag{5.8}$$

are the distributed transverse force on the string and the mass density at point x, respectively. We note, that, by virtue of the fact that the indexed position x_i is replaced by the independent spatial variable x, total derivatives with respect to the time t become partial derivatives with respect to t, whereas ratios of increments are replaced directly by partial derivatives with respect to x. Equation (5.7) represents the *partial differential equation of the string*. Similarly, conditions (5.3) must be replaced by

$$y(0,t)=y(L,t)=0\tag{5.9}$$

which are generally known as the *boundary conditions* of the problem. Equations (5.7) and (5.9) constitute what is referred to as a *boundary-value problem*. In fact, the transverse displacement $y(x,t)$ is also subject to the initial conditions

$$y(x,0)=y_0(x),\quad \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0}=v_0(x)\tag{5.10}$$

where $y_0(x)$ is the initial displacement and $v_0(x)$ the initial velocity at every point x of the string, so that Eqs.(5.7), (5.9), and (5.10) represent a *boundary-value and initial-value problem* simultaneously.

As mentioned above, the problem can be formulated more directly by considering the string as a continuous system, as shown in **Fig.5.2a**, where $f(x,t)$, $\rho(x)$ and $T(x)$ are respectively the distributed force, mass per unit length and tension at point x. **Figure 5.2b** represents the free-body diagram corresponding to an element of string of length dx. Again writing Newton's second law for the force component in the vertical direction, we obtain

$$\left[T(x)+\frac{\partial T(x)}{\partial x}dx\right]\left[\frac{\partial y(x,t)}{\partial x}+\frac{\partial^2y(x,t)}{\partial x^2}dx\right]-T(x)\frac{\partial y(x,t)}{\partial x}+f(x,t)dx=\rho(x)dx\frac{\partial^2y(x,t)}{\partial t^2}\tag{5.11}$$

Canceling appropriate terms and ignoring second-order terms in dx, Eq.(5.11) reduces to

$$\frac{\partial T(x)}{\partial x}\frac{\partial y(x,t)}{\partial x}dx+T(x)\frac{\partial^2y(x,t)}{\partial x^2}dx+f(x,t)dx=\rho(x)dx\frac{\partial^2y(x,t)}{\partial t^2}\tag{5.12}$$

and, after dividing both sides by dx, we can write Eq.(5.12) in the more compact form

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right]+f(x,t)=\rho(x)\frac{\partial^2y(x,t)}{\partial t^2}\qquad(0<x<L)\tag{5.13}$$

which is identical to Eq.(5.7) in every respect. Moreover, from **Fig.5.2b**, we recognize that the displacement of the string at the two ends must be zero, $y(0,t) = y(L,t) = 0$, thus duplicating boundary conditions (5.9). This completes the mathematical analogy between the discrete and continuous models.

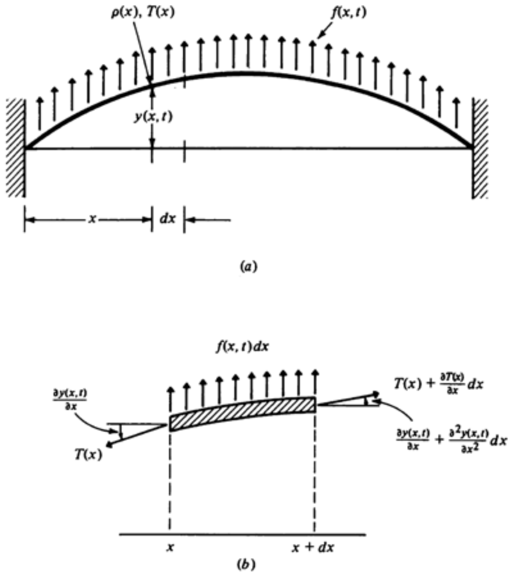


Figure 5.2

The unmistakable conclusion is that **Figs.5.1a** and **5.2a**, although different in appearance, represent two intimately related mathematical models. In this particular section we made the transition from the discrete system, **Fig.5.la**, to the continuous one, **Fig.5.2a**, through a limiting process equivalent to spreading the masses over the entire string. In many practical applications, particularly if the string is nonuniform, it is more common to follow the opposite path and lump a continuous system into discrete masses. This can be done by using the second of Eqs.(5.8) and writing $m_i = \rho(x_i)\Delta x_i$. Regardless of which mathematical model is ultimately chosen, it is clear that we should expect similar vibrational characteristics.

It turns out that the longitudinal vibration of a thin rod and the torsional vibration of a shaft of circular cross section satisfy similar boundary-value problems. In fact, the corresponding problems can be derived from the associated discrete models. For longitudinal vibration, the parameters $p(x)$ and $T(x)$ must be replaced by the mass per unit length $m(x)$ and the axial stiffness $EA(x)$, respectively, where E is the modulus of elasticity and $A(x)$ the cross-sectional area. For torsional vibration, they must be replaced by the mass polar moment of inertia per unit length $J(x)$ and the torsional stiffness $GJ(x)$, respectively, where G is the shear modulus and $J(x)$ the polar moment of inertia of the cross-sectional area.

5.3 FREE VIBRATION. THE EIGENVALUE PROBLEM

Let us consider the vibrating string of **Sec.5.2**. In the case of free vibration, namely, when the distributed force is zero, $f(x,t) = 0$, the boundary-value problem reduces to the differential equation

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right] = \rho(x)\frac{\partial^2 y(x,t)}{\partial t^2} \quad (0 < x < L) \tag{5.14}$$

and the boundary conditions

$$y(0,t) = y(L,t) = 0 \quad (5.15)$$

Although the free-vibration problem for a continuous system, Eqs.(5.14) and (5.15), differs in appearance from that of a discrete system, Eqs.(4.19), the general approach to the solution is the same. Hence, we wish to explore the possibility of synchronous motion, that is to say, a motion in which the general shape of the string displacement does not change with time, while the amplitude of this general shape does change with time. Stating it differently, every point of the string executes the same motion in time, passing through the equilibrium position at the same time and reaching its maximum excursion at the same time. In mathematical terminology, this implies that the displacement $y(x,t)$ is separable in space and time, so that we wish to examine the possibility that the solution of the boundary-value problem can be written in the form

$$y(x,t) = Y(x)F(t) \quad (5.16)$$

where $Y(x)$ represents the general string configuration and depends on the spatial variable x alone, and where $F(t)$ indicates the type of motion the string configuration executes with time and depends on t alone. Consistent with the approach used in **Sec.4.4** for discrete systems, we confine ourselves to the case in which $y(x,t)$ undergoes stable harmonic oscillation, which implies that $F(t)$ must be bounded for all times.

Introducing Eq.(5.16) into (5.14), and dividing through by $\rho(x)Y(x)F(t)$, we obtain

$$\frac{1}{\rho(x)Y(x)}\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right] = \frac{1}{F(t)}\frac{d^2F(t)}{dt^2} \quad (5.17)$$

where, because Y depends only on x and F only on t, partial derivatives have been replaced by total derivatives. Moreover, the variables have been separated so that the left side of Eq.(5.17) depends on x alone, whereas the right side depends on t alone. Using the standard argument employed in conjunction with the separation of variables method (see also **Sec.4.4**), we conclude that the only way Eq.(5.17) can be satisfied for every x and t is that both sides be constant. In view of the results derived in **Sec.4.4**, we denote the constant by $-\omega^2$, so that Eq.(5.17) leads to

$$\frac{d^2F(t)}{dt^2} + \omega^2 F(t) = 0 \quad (5.18)$$

$$-\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right] = \omega^2\rho(x)Y(x) \quad (0 < x < L) \quad (5.19)$$

We recall from **Sec.4.4** that the reason for selecting the constant as negative is for Eq.(5.18) to represent the equation of a harmonic oscillator, whose solution consists of trigonometric functions. Had we chosen a positive constant, the solution of the resulting equation would have been in terms of exponential functions, one with a positive exponent and the other with a negative one. Because the solution with the positive exponent diverges with time and that with the negative exponent decays with time, these solutions are inconsistent with the stable oscillation considered here, for which the motion amplitude must remain finite. It follows that, if synchronous motion is possible, then the function $F(t)$ expressing the time dependence must be harmonic. Hence, as in **Sec.4.4**, we can write the solution of Eq.(5.18) in the form

$$F(t) = C\cos(\omega t - \phi) \quad (5.20)$$

where C is an arbitrary constant, ω the frequency of the harmonic motion, and ϕ its phase angle, all three quantities being the same for any function $Y(x)$ that is a solution of Eq.(5.19).

The question remains as to the displacement configuration, namely, the function $Y(x)$. Clearly, $Y(x)$ must satisfy Eq.(5.19) over the domain $0 < x < L$. Moreover, from Eqs.(5.15), it must also satisfy the boundary conditions

$$Y(0) = Y(L) = 0 \quad (5.21)$$

Following is a general discussion of the solution of Eqs.(5.19) and (5.21). It parallels the discussion of the eigenvalue problem for discrete systems (see **Sec.4.4**), the various concepts being entirely analogous.

We note that Eq.(5.19) contains the parameter ω^2, as yet undetermined. The problem of determining the values of the parameter ω^2 for which nontrivial solutions $Y(x)$ of Eq.(5.19) exist, where the solutions are subject to boundary conditions (5.21), is called the *characteristic-value*, or *eigenvalue*, *problem*. The corresponding values of the parameter are known as *characteristic values*, or *eigenvalues*, and the associated functions $Y(x)$ as *characteristic functions*, or *eigenfunctions*. Equation (5.19) is a second-order ordinary differential equation and contains the parameter ω^2. Hence, we must determine two constants of integration, in addition to ω^2, but we have at our disposal only two boundary conditions. Because Eq.(5.19) is homogeneous, however, we conclude that only the shape of the function $Y(x)$ can be determined uniquely and that the amplitude of the function is arbitrary. Indeed, if $Y(x)$ is a solution of Eq. (5.19), then $\alpha Y(x)$ is also a solution, where α is a constant multiplier. It follows that one of the two boundary conditions (5.21) can be used to solve for one constant of integration in terms of the other, thus determining the general shape of $Y(x)$ but not its amplitude. The other boundary condition can be used to produce the so-called *characteristic equation*, or *frequency equation*; the values of the parameter ω^2 are obtained by solving this equation. The solution of the characteristic equation consists of a denumerably infinite set of discrete characteristic values, the square roots of which are the system *natural frequencies* ω_r $(r = 1, 2, \cdots)$. To each characteristic value, or natural frequency, corresponds an eigenfunction, or *natural mode*, $Y_r(x)$. As mentioned above, because the problem is homogeneous, $A_r Y_r(x)$ represents the same natural mode, where A_r is an arbitrary constant, so that the amplitudes of the natural modes are undetermined. The constants A_r and hence the amplitudes, can be determined uniquely if a certain normalization process is used, in which case the natural modes become *normal modes*. The natural frequencies ω_r and associated natural modes $Y_r(x)$ $(r = 1, 2, \cdots)$ depend on the system parameters $\rho(x)$ and $T(x)$, as well as on the boundary conditions; thus they are a characteristic of the system. Note that the modes $Y_r(x)$ can be regarded as infinite-dimensional eigenvectors, obtained as limiting cases of finite-dimensional eigenvectors in a process that replaces the discrete indexed position x_i by the continuous spatial variable x.

We recall that for discrete systems we also identified a set of natural frequencies and natural modes representing a characteristic of the system. Another characteristic common to discrete and continuous systems is the *orthogonality of modes*, a property to be discussed later. Hence, the analogy between discrete and continuous systems is complete, with the exception that for discrete systems the set of natural frequencies and modes is finite, whereas for continuous systems the set is infinite. The orthogonality condition can be written as

$$\int_0^L \rho(x) Y_r(x) Y_s(x) dx = 0 \qquad (r \neq s) \tag{5.22}$$

where $Y_r(x)$ and $Y_s(x)$ are two distinct eigenfunctions. For convenience, the modes can be normalized by writing

$$\int_0^L \rho(x) Y_r(x) Y_s(x) dx = \delta_{rs} \qquad (r, s = 1, 2, \cdots) \tag{5.23}$$

where δ_{rs} is the Kronecker delta. Moreover, we shall verify later that the eigenfunctions $Y_r(x)$ satisfy also the relation

$$\int_0^L T(x) \frac{dY_r(x)}{dx} \frac{dY_s(x)}{dx} dx = \omega_r^2 \delta_{rs} \qquad (r, s = 1, 2, \cdots) \tag{5.24}$$

In view of the above, the free-vibration solution of Eq.(5.14) can be represented by an infinite series of the system eigenfunctions in the form

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(x) \quad (5.25)$$

Introducing Eq.(5.25) into (5.14), multiplying the result by $Y_s(x)$, integrating over the domain $0 < x < L$, recalling that the system eigenfunctions satisfy Eq.(5.19) and assuming that they are normalized so as to satisfy conditions (5.23) and (5.24), we arrive at the infinite set of harmonic equations

$$\ddot{\eta}_r(t) + \omega_r{}^2\eta_r(t) = 0 \quad (r = 1, 2, \cdots) \quad (5.26)$$

where the time-dependent functions $\eta_r(t)$ are the system *natural coordinates*, which in this case are also *normal coordinates*. As in **Sec.4.4**, the solution of Eqs.(5.26) can be written as

$$\eta_r(t) = C_r \cos(\omega_r t - \phi_r) \quad (r = 1, 2, \cdots) \quad (5.27)$$

where the constants C_r and ϕ_r are the amplitude and phase angle, respectively, quantities which depend on the initial conditions. The response of the system to initial conditions can be obtained by inserting (5.27) into (5.25). We shall not pursue the subject any further at this point, but return to it in **Sec.5.9**, where the response to both initial excitation and forcing functions is presented.

A simple illustration of the solution of the eigenvalue problem is furnished in **Example 5.1**. Further elaboration, including a proof of the orthogonality property, is provided in subsequent sections.

Example 5.1

Solve the eigenvalue problem associated with a uniform string fixed at $x = 0$ and $x = L$ (see **Fig.5.3**), and plot the first three eigenfunctions. The tension T in the string is constant.

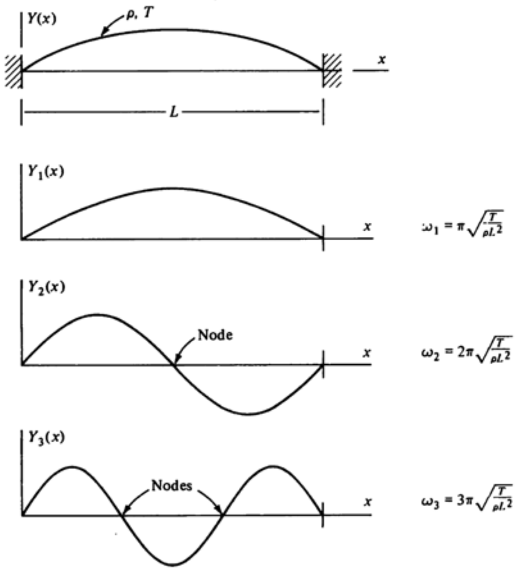


Figure 5.3

Inserting $\rho(x) = \rho = \text{const}$, $T(x) = T = \text{const}$ in Eq.(5.19), we conclude that the transverse displacement $Y(x)$ must satisfy the differential equation

$$\frac{d^2Y(x)}{dx^2} + \beta^2 Y(x) = 0 \quad \left(\beta^2 = \frac{\omega^2\rho}{T}\right) \quad (a)$$

over the domain $0 < x < L$. Moreover, because the ends are fixed, the displacement must be zero at $x = 0$ and $x = L$.

Hence, the solution Y of Eq. (a) is subject to the boundary conditions

$$Y(0)=0, \quad Y(L)=0 \tag{b}$$

Equation (a) is harmonic in x, and its solution can be written in the form

$$Y(x)=A\sin\beta x+\cos\beta x \tag{c}$$

where A and B are constants of integration. Inserting the first of boundary conditions (b) into (c), we conclude that $B=0$, so that the solution reduces to

$$Y(x)=A\sin\beta x \tag{d}$$

On the other hand, introducing the second of boundary conditions (b) into Eq.(d), we obtain

$$Y(L)=A\sin\beta L=0 \tag{e}$$

There are two ways in which Eq.(e) can be satisfied, namely, $A=0$ and $\sin\beta L=0$. But $A=0$ must be ruled out, because this would yield the trivial solution $Y(x)=0$. It follows that we must have

$$\sin\beta L=0 \tag{f}$$

which is recognized as the *characteristic equation*. Its solution consists of the infinite set of *characteristic values*.

$$\beta_r L=r\pi \qquad (r=1,2,\cdots) \tag{g}$$

to which corresponds the infinite set of *eigenfunctions*

$$Y_r(x)=A_r\sin\frac{r\pi x}{L} \tag{h}$$

where A_r are undetermined amplitudes, with the implication that only the mode shapes can be determined uniquely. The first three natural modes are plotted in **Fig.5.3**, where the modes have been normalized by letting $A_r=1$. We note that the first mode has no nodes, the second has one node and the third has two nodes. In general the r-th mode has $r-1$ nodes $(r=1,2,\cdots)$.

From the second of Eqs.(a) we conclude that the system natural frequencies are

$$\omega_r=\beta_r\sqrt{\frac{T}{\rho}}=r\pi\sqrt{\frac{T}{\rho L^2}} \qquad (r=1,2,\cdots) \tag{i}$$

The frequency ω_1 is called the *fundamental frequency* and the higher frequencies ω_r $(r=2,3,\cdots)$ are referred to as *overtones*. The overtones are integral multiples of the fundamental frequency, for which reason the fundamental frequency is called the *fundamental harmonic* and the overtones are known as *higher harmonics*.

Vibrating systems which possess harmonic overtones are distinguished by the fact that under certain excitations they produce pleasant sounds. Such systems are not commonly encountered in nature but can be manufactured, particularly for use in musical instruments. It is a well-known fact that the string is the major ingredient in a large number of musical instruments, such as the violin, the piano, the guitar and many other instruments related to them. For example, the violin has four strings which possess four fundamental frequencies. From Eq.(i), we observe that these frequencies depend on the tension T, the mass density ρ and the length L. The violinist tuning a violin merely ensures that the strings have the proper tension. This is done by comparing the pitch of a given note to that produced by a different instrument known to be tuned correctly. One must not infer from this, however, that the violin yields only four fundamental frequencies and their higher harmonics. Indeed, whereas ρ and T are constant for each string, the violinist can change the pitch by adjusting the length of the strings. Hence, when fingers are run on the fingerboard, the artist merely adjusts the length L of the strings. Thus, there is a large variety of frequencies at the violinist's disposal. Generally the sounds consist of a combination of harmonics, with the lower harmonics being the predominant ones. However, a talented performer excites the proper array of higher harmonics to produce a pleasing sound.

Example 5.2

Consider the eigenvalue problem of **Example 5.1** and verify that the eigenfunctions satisfy the orthogonality relations, Eqs.(5.23) and (5.24).

Before verifying the satisfaction of Eqs.(5.23) and (5.24), we must normalize the modes according to

$$\int_0^L \rho(x)Y_r^2(x)dx = 1 \qquad (r = 1, 2, \cdots) \tag{a}$$

Hence, inserting Eqs.(h) of **Example 5.1** into Eq.(a), and recalling that $\rho(x) = \rho = \text{const}$, we can write

$$\rho A_r^2 \int_0^L \sin^2 \frac{r\pi x}{L} dx = 1 \qquad (r = 1, 2, \cdots) \tag{b}$$

But $\sin^2 \alpha = (1 - \cos 2\alpha)/2$, so that

$$\int_0^L \sin^2 \frac{r\pi x}{L} dx = \frac{1}{2}\int_0^L \left(1 - \cos\frac{2r\pi x}{L}\right)dx = \frac{L}{2} \tag{c}$$

Inserting Eq.(c) into (b), we conclude that

$$A_r = \sqrt{\frac{2}{\rho L}} \qquad (r = 1, 2, \cdots) \tag{d}$$

so that the normal modes become

$$Y_r(x) = \sqrt{\frac{2}{\rho L}} \sin\frac{r\pi x}{L} \qquad (r = 1, 2, \cdots) \tag{e}$$

Using Eq.(e), we can form

$$\int_0^L \rho Y_r(x)Y_s(x)dx = \frac{2}{L}\int_0^L \rho \sin\frac{r\pi x}{L}\sin\frac{s\pi x}{L}dx \tag{f}$$

Recalling that $\sin\alpha \sin\beta = \{\cos(\alpha - \phi) - \cos(\alpha + \beta)\}/2$, we can write

$$\int_0^L \sin\frac{r\pi x}{L}\sin\frac{s\pi x}{L}dx = \frac{1}{2}\int_0^L \left[\cos\frac{(r-s)\pi x}{L} - \cos\frac{(r+s)\pi x}{L}\right]dx = \begin{cases} 0 & r \neq s \\ L/2 & r = s \end{cases} \tag{g}$$

Hence, inserting Eq.(g) into (f), we can write

$$\int_0^L \rho Y_r(x)Y_s(x)dx = \delta_{rs} \qquad (r, s = 1, 2, \cdots) \tag{h}$$

where δ_{rs} is the Kronecker delta, thus verifying Eq.(5.23).

To verify Eq.(5.24), we follow a procedure similar to that above, recall Eq.(i) of **Example 5.1**, and write

$$\int_0^L T(x)\frac{dY_r(x)}{dx}\frac{dY_s(x)}{dx}dx = T\frac{2}{\rho L}\frac{r\pi}{L}\frac{s\pi}{L}\int_0^L \cos\frac{r\pi x}{L}\cos\frac{s\pi x}{L}dx = \omega_r^2\delta_{rs} \qquad (r, s = 1, 2, \cdots) \tag{i}$$

which is identical to Eq.(5.24).

Note that the fact that the eigenfunctions $Y_r(x)$ in this particular case satisfy relations (5.23) and (5.24) is a mere reiteration of the ordinary orthogonality of trigonometric functions. However, we shall have the opportunity to establish that the orthogonality of the eigenfunctions is much more general in nature, as the eigenfunctions of a system are trigonometric functions only in very special cases.

5.4 CONTINUOUS VERSUS DISCRETE MODELS FOR THE AXIAL VIBRATION OF RODS

To bring the parallel between continuous and discrete models into sharper focus, we consider a specific system and compare the solutions of the eigenvalue problem obtained by regarding the same system first as continuous and then as discrete. A system that lends itself readily to such an analysis is the rod in axial vibration.

As indicated in **Sec.5.2**, the boundary-value problem for the axial vibration of a thin rod has the same structure as that for the transverse vibration of a string. To obtain the first from the second, we must replace the system parameters $\rho(x)$ and $T(x)$ by $m(x)$ and $EA(x)$, respectively, where $m(x)$ is the mass per unit length of rod and $EA(x)$ the axial stiffness, in which E is the modulus of elasticity and $A(x)$ the cross-sectional area. It also follows that the structure of the eigenvalue problems is similar, subject to the same parameter substitution.

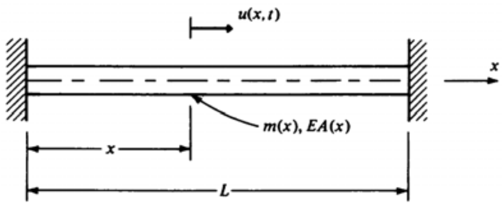


Figure 5.4

Let us consider the axial vibration of a thin rod fixed at both ends (see **Fig.5.4**). In view of the above discussion, if we use Eqs.(5.19) and (5.21) and assume that the axial displacement $u(x,t)$ is separable in space and time, or

$$u(x,t) = U(x)F(t) \tag{5.28}$$

in which $F(t)$ is harmonic, we can write the eigenvalue problem directly in the form

$$-\frac{d}{dx}\left[EA(x)\frac{dU(x)}{dx}\right] = \omega^2 m(x)U(x) \quad (0 < x < L) \tag{5.29}$$

where $U(x)$ is subject to the boundary conditions

$$U(0) = U(L) = 0 \tag{5.30}$$

The differential equation (5.29) possesses space-dependent coefficients, so that in general no closed-form solution can be expected. A closed-form solution can be obtained in the special case of a *uniform rod*, $m(x) = m = \text{const}$, $EA(x) = EA = \text{const}$. Considering that case, Eq.(5.29) reduces to

$$\frac{d^2U(x)}{dx^2} + \beta^2 U(x) = 0 \qquad \beta^2 = \omega^2\frac{m}{EA} \tag{5.31}$$

which must be satisfied over the domain $0 < x < L$. Of course, boundary conditions (5.30) remain the same.

The eigenvalue problem defined by the differential equation (5.31) and the boundary conditions (5.30) has precisely the same structure as that for the string fixed at both ends discussed in **Example 5.1**. It follows that the solution has the same structure, subject to the parameter substitution pointed out above. Hence, using the results of **Example 5.1**, we can write directly the system natural frequencies

$$\omega_r = \beta_r\sqrt{\frac{EA}{m}} = r\pi\sqrt{\frac{EA}{mL^2}} \qquad (r = 1, 2, \cdots) \tag{5.32}$$

Moreover, if the modes are normalized by letting $A_r = 1 \quad (r = 1, 2, \cdots)$, we obtain the normal modes

$$U_r(x) = \sin\frac{r\pi x}{L} \qquad (r = 1, 2, \cdots) \tag{5.33}$$

The first five normal modes are plotted in **Fig.5.5** in solid lines.

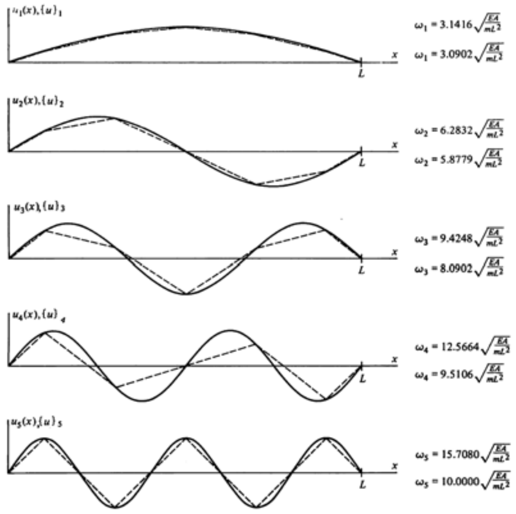


Figure 5.5

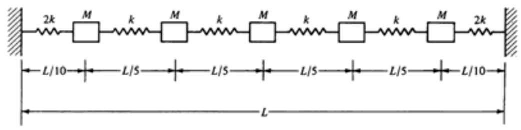


Figure 5.6

Next, let us solve the same problem by regarding the system as discrete. An equivalent discrete system can be obtained by dividing the rod into five equal segments, lumping the mass of the segments in the center as shown in **Fig.5.6** and regarding the lumped masses M as being connected by springs of equivalent stiffnesses k and $2k$, where k is such that the springs undergo the same elongations as the corresponding rod segments would under identical loading. Hence, the lumped masses have the value $M = mL/5$ and the spring constant is $k = 5EA/L$. Accordingly, the eigenvalue problem can be written as

$$[k]\{u\} = \omega^2 [m]\{u\} \tag{5.34}$$

where the stiffness matrix has the form

$$[k] = \frac{5EA}{L} \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \tag{5.35}$$

whereas the mass matrix is simply

$$[m] = \frac{mL}{5} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.36}$$

The solution of the eigenvalue problem (5.34), in conjunction with matrices (5.35) and (5.36), was obtained by means of a computer program, with the results

$$\{u\}_1 = \begin{Bmatrix} 0.3090 \\ 0.8090 \\ 1.0000 \\ 0.8090 \\ 0.3090 \end{Bmatrix}, \quad \omega_1 = 3.0902\sqrt{\frac{EA}{mL^2}} \tag{5.37a}$$

$$\{u\}_2 = \begin{Bmatrix} 0.5878 \\ 0.9511 \\ 0 \\ -0.9511 \\ -0.5878 \end{Bmatrix}, \quad \omega_2 = 5.8779\sqrt{\frac{EA}{mL^2}} \tag{5.37b}$$

$$\{u\}_3 = \begin{Bmatrix} 0.8090 \\ 0.3090 \\ -1.0000 \\ 0.3090 \\ 0.8090 \end{Bmatrix}, \quad \omega_3 = 8.0902\sqrt{\frac{EA}{mL^2}} \tag{5.37c}$$

$$\{u\}_4 = \begin{Bmatrix} 0.9511 \\ -0.5878 \\ 0 \\ 0.5878 \\ -0.9511 \end{Bmatrix}, \quad \omega_4 = 9.5106\sqrt{\frac{EA}{mL^2}} \tag{5.37d}$$

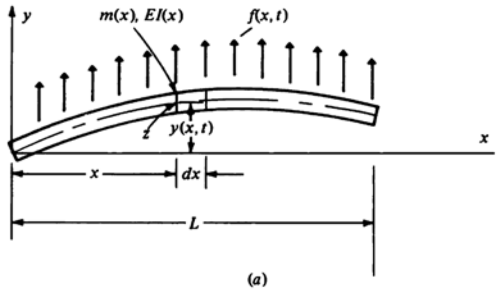
$$\{u\}_5 = \begin{Bmatrix} 1.0000 \\ -1.0000 \\ 1.0000 \\ -1.0000 \\ 1.0000 \end{Bmatrix}, \quad \omega_5 = 10.0000\sqrt{\frac{EA}{mL^2}} \tag{5.37e}$$

where the modes have been normalized so as to match the amplitudes of the normal modes (5.33) of the continuous system. The modes are plotted in **Fig.5.5** in dashed lines. The natural frequencies of the continuous model are given on the corresponding top lines and those of the discrete model on the bottom lines. It is easy to see that, whereas the first mode and natural frequency are relatively close to those of the continuous model, accuracy is lost rapidly for higher modes in the discrete model, in the sense that the displacements are not very representative and the frequencies are not good approximations of those of the continuous system.

We note that the natural frequencies of the discrete system are lower than those of the corresponding continuous model. The reason is that, although the total mass is the same in both systems, in the case of the discrete model the mass is shifted toward the center of the system instead of being uniformly distributed. This tends to increase the effect of the system inertia relative to its stiffness, resulting in lower natural frequencies. Of course, accuracy can be improved by increasing the number of degrees of freedom of the discrete system.

5.5 BENDING VIBRATION OF BARS. BOUNDARY CONDITIONS

The transverse vibration of a string, axial vibration of a thin rod and torsional vibration of a circular shaft all lead to the same form of boundary-value problem, namely, one consisting of a partial differential equation of second order in both space and time and two boundary conditions, one at each end. Of course, the system parameters are different in each case (see **Sec.5.2**). By contrast, the boundary-value problem for a bar in flexure is defined by a fourth-order differential equation in space requiring two boundary conditions at each end. In this section, we derive this boundary-value problem and use the opportunity to discuss the nature of various types of boundary conditions.



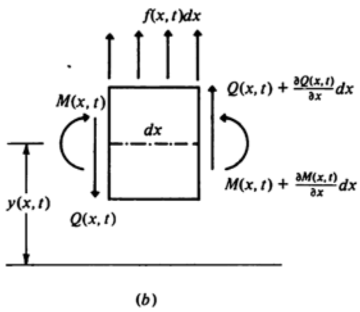


Figure 5.7

Let us consider the bar in flexure shown in **Fig.5.1a**. The transverse displacement at any point x and time t is denoted by $y(x,t)$ and the transverse force per unit length by $f(x,t)$. The system parameters are the mass per unit length $m(x)$ and the flexural rigidity $EI(x)$, where E is Young's modulus of elasticity and $I(x)$ the cross-sectional area moment of inertia about an axis normal to x and y and passing through the center of the cross-sectional area. **Figure 5.7b** shows the free-body diagram corresponding to a bar element of length dx, where $Q(x,t)$ denotes the shearing force and $M(x,t)$ the bending moment. We use the so-called "simple-beam theory," according to which the rotation of the element is insignificant compared to the vertical translation, and the shear deformation is small in relation to the bending deformation. This theory is valid if the ratio between the length of the bar and its height is relatively large (say more than 10), and if the bar does not become too "wrinkled" because of flexure. In the area of vibrations the above statements imply ignoring the rotatory inertia and shear deformation effects.

From **Fig.5.7b**, the force equation of motion in the vertical direction has the form

$$\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]-Q(x,t)+f(x,t)dx=m(x)dx\frac{\partial^2 y(x,t)}{\partial t^2} \tag{5.38}$$

On the other hand, ignoring the inertia torque associated with the rotation of the element, the moment equation of motion about the axis normal to x and y and passing through the center of the cross-sectional area is

$$\left[M(x,t)+\frac{\partial M(x,t)}{\partial x}dx\right]-M(x,t)+\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]dx+f(x,t)dx\frac{dx}{2}=0 \tag{5.39}$$

Canceling appropriate terms and ignoring terms involving second powers in dx, we can write Eq.(5.39) in the simple form

$$\frac{\partial M(x,t)}{\partial x} + Q(x,t) = 0 \tag{5.40}$$

Moreover, canceling appropriate terms and considering (5.40), Eq.(5.38) reduces to

$$-\frac{\partial^2 M(x,t)}{\partial x^2} + f(x,t) = m(x)\frac{\partial^2 y(x,t)}{\partial t^2} \tag{5.41}$$

which must be satisfied over the domain $0 < x < L$.

Equation (5.41) relates the bending moment $M(x,t)$, the transverse force $f(x,t)$ and the bending displacement $y(x,t)$. Any elementary text on mechanics of materials, however, gives the relation between the bending moment and bending deformation in the form

$$M(x,t) = EI(x)\frac{\partial^2 y(x,t)}{\partial x^2} \tag{5.42}$$

Inserting Eq.(5.42) into (5.41), we obtain the differential equation for the flexural vibration of a bar

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] + f(x,t) = m(x)\frac{\partial^2 y(x,t)}{\partial t^2} \quad (0 < x < L) \tag{5.43}$$

where we note that the equation contains spatial derivatives through fourth order.

To complete the formulation of the boundary-value problem, we must specify the boundary conditions. We list here the most common ones:

1. *Clamped end* at $x = 0$. The deflection and slope of the deflection curve are zero:

$$y(0,t) = 0\,, \quad \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0} = 0 \tag{5.44}$$

2. *Hinged end* at $x = 0$. The deflection and bending moment are zero:

$$y(0,t) = 0\,, \quad \left.EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0} = 0 \tag{5.45}$$

Note that Eq.(5.42) was used in the second of conditions (5.45).

3. *Free end* at $x = 0$. The bending moment and shearing force are zero. Using Eqs.(5.40) and (5.42), the boundary conditions become

$$\left.EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0} = 0\,, \quad \left.\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]\right|_{x=0} = 0 \qquad (5.46),$$

Analogous conditions can be written for the end $x = L$. Of course, there are less common boundary conditions, such as when the end is supported by springs, or when there is a concentrated mass at the end.

At this point a discussion of the character of the boundary conditions is in order. It is worth noting that boundary conditions (5.44) and the first of (5.45) are a result of the system geometry. For this reason they are called *geometric boundary conditions*. On the other hand, the second of boundary conditions (5.45) and both of (5.46) reflect the force and moment balance at the boundary; they are called *natural boundary conditions*.

Turning our attention to the corresponding eigenvalue problem, we first consider the free vibration characterized by $f(x,t) = 0$, in which case the solution of Eq. (5.43) becomes separable in space and time. Letting

$$y(x,t) = Y(x)F(t) \tag{5.47}$$

and using the separation of variables method, as in **Sec.5.3**, it can be shown that $F(t)$ is harmonic in this case also. This is no coincidence, however, as for all the conservative systems discussed here the time dependence is harmonic.

Denoting the frequency of $F(t)$ by ω, the eigenvalue problem formulation reduces to the differential equation

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]=\omega^2 m(x)Y(x) \quad (0<x<L) \tag{5.48}$$

where the function $Y(x)$ must satisfy appropriate boundary conditions. Inserting Eq.(5.47) into (5.44) through (5.46), and eliminating the time dependence, we obtain boundary conditions similar in form to (5.44) through (5.46), with the exception that $y(x,t)$ is replaced by $Y(x)$ and partial derivatives with respect to x by total derivatives with respect to x.

5.6 NATURAL MODES OF A BAR IN BENDING VIBRATION

It should be clear by now that, to obtain the natural modes of a system, we must solve an eigenvalue problem. The eigenvalue problem associated with a bar in bending vibration, as derived in **Sec.5.5**, consists of the differential equation

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]=\omega^2 m(x)Y(x) \quad (0<x<L) \tag{5.49}$$

where $EI(x)$ is the flexural rigidity and $m(x)$ the mass per unit length at any point x. The solution $Y(x)$ is subject to given boundary conditions reflecting the manner in which the ends are supported. Several examples of boundary conditions can be obtained by replacing $y(x,t)$ by $Y(x)$ and partial derivatives by total derivatives with respect to x in Eqs.(5.44)-(5.46). Equation (5.49) possesses coefficients depending on the spatial variable and has no general closed-form solution. Solutions can be obtained for certain special cases, most notably those in which the bar is uniform.

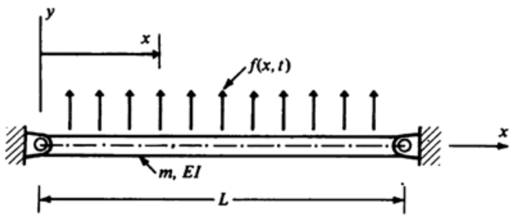


Figure 5.8

Let us consider the uniform bar hinged at both ends shown in **Fig.5.8**, for which the differential equation (5.49) reduces to

$$\frac{d^4Y(x)}{dx^4}-\beta^4 Y(x)=0\,, \quad \beta^4=\frac{\omega^2 m}{EI} \tag{5.50}$$

where EI and m are constant. The boundary conditions are obtained from Eqs.(5.45). Indeed, at the end $x=0$, the boundary conditions are

$$Y(0)=0\,, \quad \left.\frac{d^2Y(x)}{dx^2}\right|_{x=0}=0 \tag{5.51}$$

whereas at the end $x=L$, the boundary conditions are

$$Y(L)=0\,, \quad \left.\frac{d^2Y(x)}{dx^2}\right|_{x=L}=0 \tag{5.52}$$

We note that the first boundary condition in both (5.51) and (5.52) is geometric and the second is natural.

The general solution of Eq.(5.50) can be easily verified to be

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \quad (5.53)$$

where $C_i \quad (i = 1, 2, 3, 4)$ are constants of integration. To evaluate three of these constants in terms of the fourth, as well as to derive the characteristic equation, we must use boundary conditions (5.51) and (5.52). Indeed, the first of boundary conditions (5.51) yields $C_2 + C_4 = 0$, whereas the second of (5.51) gives $-C_2 + C_4 = 0$, with the obvious conclusion that

$$C_2 = C_4 = 0 \quad (5.54)$$

Hence, solution (5.53) reduces to

$$Y(x) = C_1 \sin \beta x + C_3 \sinh \beta x \quad (5.55)$$

On the other hand, boundary conditions (5.52) lead to the two simultaneous equations

$$\begin{aligned} C_1 \sin \beta L + C_3 \sinh \beta L = 0 \\ -C_1 \sin \beta L + C_3 \sinh \beta L = 0 \end{aligned} \quad (5.56)$$

yielding

$$C_3 = 0 \quad (5.57)$$

and the characteristic equation

$$\sin \beta L = 0 \quad (5.58)$$

There are two other solutions of Eqs.(5.56), namely $C_1 = 0$, $\sinh \beta L = 0$ and $C_1 = C_3 = 0$, but they represent trivial solutions.

The solution of the characteristic equation is simply

$$\beta_r L = r\pi \quad (5.59)$$

yielding the natural frequencies

$$\omega_r = (r\pi)^2 \sqrt{\frac{EI}{mL^4}} \quad (r = 1, 2, \cdots) \quad (5.60)$$

Moreover, recalling that $C_3 = 0$, using the values of $\beta_r \quad (r = 1, 2, \cdots)$ given by Eq.(5.59) and normalizing according to $\int_0^L mY_r^2(x)dx = 1 \quad (r = 1, 2, \cdots)$, we obtain the normal modes

$$Y_r(x) = \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L} \quad (r = 1, 2, \cdots) \quad (5.61)$$

The first three modes are like those plotted in **Fig.5.3** but the frequencies are different. Note that the number of nodes is equal to the mode number minus 1.

Next let us consider the *clamped-free uniform bar* of **Fig.5.9**. While the differential equation remains in the from (5.50), the boundary conditions at the clamped end, $x = 0$, are

$$Y(0) = 0, \quad \left.\frac{dY(x)}{dx}\right|_{x=0} = 0 \quad (5.62)$$

On the other hand, at the free end, $x = L$, the boundary conditions reduce to

$$\left.\frac{d^2Y(x)}{dx^2}\right|_{x=L} = 0, \quad \left.\frac{d^3Y(x)}{dx^3}\right|_{x=L} = 0 \quad (5.63)$$

We note that boundary conditions (5.62) and (5.63) are geometric and natural, respectively.

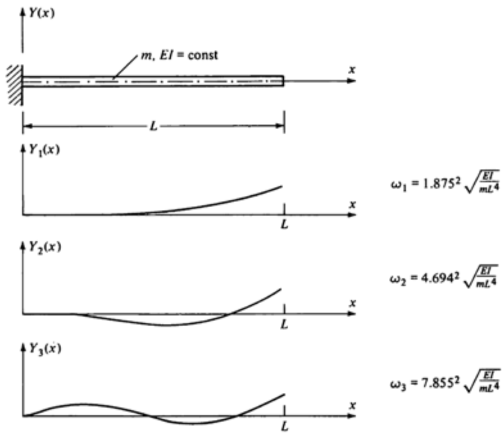


Figure 5.9

The general solution remains in the form (5.53), but the constants C_i $(i=1,2,3,4)$ have different values. While the first of boundary conditions (5.62) leads once again to $C_2+C_4=0$, the second of (5.62) yields $C_1+C_3=0$, so that solution (5.53) takes the form

$$Y(x)=C_1(\sin\beta x-\sinh\beta x)+C_2(\cos\beta x-\cosh\beta x) \tag{5.64}$$

Using boundary conditions (5.63), we arrive at the two simultaneous equations

$$C_1(\sin\beta L+\sinh\beta L)+C_2(\cos\beta L+\cosh\beta L)=0 \tag{5.65}$$

$$C_1(\cos\beta L+\cosh\beta L)-C_2(\sin\beta L-\sinh\beta L)=0 \tag{5.66}$$

Equation (5.66) can be solved for C_2 in terms of C_1 and the result inserted into (5.64) and (5.65) to yield

$$Y(x)=\frac{C_1}{\sin\beta L-\sinh\beta L}\left[(\sin\beta L-\sinh\beta L)(\sin\beta x-\sinh\beta x)+(\cos\beta L+\cosh\beta L)(\cos\beta x-\cosh\beta x)\right] \tag{5.67}$$

and

$$C_1\left[(\sin\beta L+\sinh\beta L)(\sin\beta L-\sinh\beta L)+(\cos\beta L+\cosh\beta L)^2\right]=0 \tag{5.68}$$

Because for a nontrivial solution we must have $C_1\neq 0$, the expression inside the brackets in (5.68) must be zero. After simplification, this leads to the characteristic equation

$$\cos\beta L\cosh\beta L=-1 \tag{5.69}$$

The solution of Eq.(5.69) must be obtained numerically, yielding an infinite set of eigenvalues β_r $(r=1,2,\cdots)$. Inserting these values into Eq.(5.67), we obtain the natural modes

$$Y_r(x)=A_r\left[(\sin\beta_r L-\sinh\beta_r L)(\sin\beta_r x-\sinh\beta_r x)+(\cos\beta_r L+\cosh\beta_r L)(\cos\beta_r x-\cosh\beta_r x)\right]\quad (r=1,2,\cdots) \tag{5.70}$$

where the notation $A_r=C_1/(\sin\beta_r L-\sinh\beta_r L)$ has been introduced for simplicity. The first three modes are plotted in **Fig.5.9**, and we note once again that the mode $Y_r(x)$ has $r-1$ nodes $(r=1,2,\cdots)$.

Although the characteristic equation, Eq. (5.58) or Eq. (5.69), yields an infinity of characteristic values leading to the associated natural modes, we should recall that, because of the simple-beam theory limitations, the higher modes become increasingly inaccurate. This is so because the number of nodes increases with each mode, so that the distance between nodes decreases accordingly and the bar becomes progressively more "wrinkled." Hence, as the mode number increases, the rotation of a bar element can no longer be considered negligible compared with the translation, so that the simple-beam theory is not valid for the very high modes.

5.7 ORTHOGONALITY OF NATURAL MODES. EXPANSION THEOREM

In **Sec.5.3**, it was mentioned that the eigenfunctions are orthogonal in a manner similar to the way in which the eigenvectors for discrete systems are. In fact, we shall prove the orthogonality property by first regarding the system as discrete and taking the limit, and then working directly with the continuous system.

Considering an n-degree-of-freedom discrete system, we have shown in **Sec.4.5** that two eigenvectors $\{u\}_r$ and $\{u\}_s$ corresponding to distinct eigenvalues ω_r^2 and ω_s^2 are orthogonal with respect to the mass matrix. Without loss of generality, we can assume that the mass matrix is diagonal, so that Eq.(4.80) can be written as the sum

$$\sum_{i=1}^{n} m_i u_{ir} u_{is} = 0 \quad (r \neq s) \tag{5.71}$$

where m_i is the mass in the position $x = x_i$ and u_{ir} and u_{is} are the displacements of m_i in the modes r and s, respectively. Following a pattern similar to that used in **Sec.5.2**, we can increase the number n of masses m_i indefinitely, while reducing the size of the masses and the distance between any two masses, so that relations of the form

$$m_i = \rho(x_i)\Delta x_i \quad (r = 1, 2, \cdots) \tag{5.72}$$

are preserved, where $\rho(x_i)$ is an equivalent mass per unit length at the point $x = x_i$. Inserting Eqs.(5.72) into (5.71), we obtain

$$\sum_{i=1}^{n} \rho(x_i) u_{ir} u_{is} \Delta x_i = 0 \quad (r \neq s) \tag{5.73}$$

In the limit, as $\Delta x_i \to 0$, we can replace the indexed variable x_i by the continuous independent variable x, so that the sum reduces to the integral

$$\int_0^L \rho(x) u_r(x) u_s(x) dx = 0 \quad (r \neq s) \tag{5.74}$$

where $u_r(x)$ and $u_s(x)$ are the eigenfunctions obtained by letting the number of components of the eigenvectors $\{u\}_r$ and $\{u\}_s$ increase indefinitely. Equation (5.74) implies that the eigenfunctions $u_r(x)$ and $u_s(x)$ are orthogonal with respect to the mass density $\rho(x)$.

The orthogonality of eigenfunctions can be proved in a very general way, without the explicit knowledge of the eigenfunctions, by using operator notation. However, because our text is more limited in scope, we would like to dispense with operator notation. Nevertheless, we can still use the idea of proving orthogonality for a given set of eigenfunctions without actually solving the eigenvalue problem. To this end, we consider the eigenvalue problem given by Eq.(5.48), subject to appropriate boundary conditions. Denoting two distinct solutions of the eigenvalue problem by $Y_r(x)$ and $Y_s(x)$, respectively, we can write

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right] = \omega_r^2 m(x) Y_r(x) \quad 0 < x < L \tag{5.75}$$

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_s(x)}{dx^2}\right] = \omega_s^2 m(x) Y_s(x) \quad 0 < x < L \tag{5.76}$$

Next let us multiply Eq.(5.75) through by $Y_s(x)$, and integrate by parts over the domain $0 < x < L$, to obtain

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx = \left\{Y_s(x)\frac{d}{dx}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\right\}\Bigg|_0^L - \left[\frac{dY_s(x)}{dx}EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\Bigg|_0^L$$
$$+\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx = \omega_r^2\int_0^L m(x)Y_r(x)Y_s(x)dx \tag{5.77a}$$

Multiplying Eq.(5.76) through by $Y_r(x)$, and performing a similar integration by parts, we arrive at

$$\int_0^L Y_r(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_s(x)}{dx^2}\right]dx = \left\{Y_r(x)\frac{d}{dx}\left[EI(x)\frac{d^2Y_s(x)}{dx^2}\right]\right\}\Bigg|_0^L - \left[\frac{dY_r(x)}{dx}EI(x)\frac{d^2Y_s(x)}{dx^2}\right]\Bigg|_0^L$$
$$+\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx = \omega_s^2\int_0^L m(x)Y_r(x)Y_s(x)dx \tag{5.77b}$$

Subtracting Eq.(5.77b) from (5.77a), we obtain

$$\left(\omega_r^2-\omega_s^2\right)\int_0^L m(x)Y_r(x)Y_s(x)dx$$
$$=\left\{Y_s(x)\frac{d}{dx}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\right\}\Bigg|_0^L - \left[\frac{dY_s(x)}{dx}EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\Bigg|_0^L \tag{5.78}$$
$$-\left\{Y_r(x)\frac{d}{dx}\left[EI(x)\frac{d^2Y_s(x)}{dx^2}\right]\right\}\Bigg|_0^L - \left[\frac{dY_r(x)}{dx}EI(x)\frac{d^2Y_s(x)}{dx^2}\right]\Bigg|_0^L$$

We shall consider only those systems for which the end conditions are such that the right side of (5.78) vanishes. Clearly, this is the case when the system has any combination of clamped, hinged, and free ends, as can be concluded from **Sec.5.5**. It can be shown that the right side of (5.78) is zero also when the ends are supported by means of springs. Hence, Eq.(5.78) reduces to

$$\left(\omega_r^2-\omega_s^2\right)\int_0^L m(x)Y_r(x)Y_s(x)dx = 0 \tag{5.79}$$

But, according to our assumption $Y_r(x)$ and $Y_s(x)$ are eigenfunctions corresponding to distinct eigenvalues, $\omega_r^2 \neq \omega_s^2$ for $r \neq s$. It follows that

$$\int_0^L m(x)Y_r(x)Y_s(x)dx = 0 \qquad (r \neq s) \tag{5.80}$$

so that the eigenfunctions $Y_r(x)$ and $Y_s(x)$ are orthogonal with respect to the mass density $m(x)$. We note the complete analogy with Eq.(5.74), where the latter was derived as a limiting case of a discrete system.

While the eigenvectors $u_r(x)$ associated with a discrete system are also orthogonal with respect to the stiffness matrix, as stated by Eq.(4.81), the eigenfunctions are orthogonal with respect to the stiffness $EI(x)$ only in a certain sense. To explain the meaning of this statement, let us multiply Eq.(5.75) by $Y_s(x)$ and integrate over the length of the bar, so that

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx = \omega_r^2\int_0^L m(x)Y_r(x)Y_s(x)dx \tag{5.81}$$

In view of Eq.(5.80), however, we can write

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx = 0 \qquad (r \neq s) \tag{5.82}$$

so that the eigenfunctions are orthogonal with respect to the stiffness $EI(x)$ in the sense indicated by Eq.(5.82). Equation (5.82) can be shown to lead to a more convenient form. Indeed, integrating the equation by parts, we obtain

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx=\left\{Y_s(x)\frac{d}{dx}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\right\}\Bigg|_0^L-\left[\frac{dY_s(x)}{dx}EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\Bigg|_0^L \qquad (r\neq s)$$
$$+\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx=0$$

(5.83)

If the boundary conditions are as stipulated above, then we have

$$\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx=0 \qquad (r\neq s) \tag{5.84}$$

so that the second derivatives of the eigenfunctions, but not the eigenfunctions themselves, are orthogonal with respect to the stiffness $EI(x)$. Note that the order of the derivatives involved in the orthogonality condition (5.84) is related to the order of the eigenvalue problem. Indeed, the order of the derivatives is equal to one-half the order of the eigenvalue problem. This can be explained easily by the fact that Eq.(5.84) is obtained from Eq.(5.82) through integrations by parts. Because the sum of the order of the highest derivatives of $Y_s(x)$ and $Y_r(x)$ in the two equations must be the same, and the order of the derivative of Y_s in Eq.(5.82) is zero, it follows that the order of the derivatives of $Y_r(x)$ and $Y_s(x)$ in Eq.(5.84) is one-half the order of the highest derivative of Y_r in Eq.(5.82), where the highest derivative determines the order of the eigenvalue problem.

When $r=s$ the integral in Eq.(5.80) is a positive quantity except in the case of the trivial solution, which presents no interest. Recalling that the eigenvalue problem is homogeneous, we can normalize the natural modes by writing

$$\int_0^L m(x)Y_r(x)Y_s(x)dx=\delta_{rs} \qquad (r,s=1,2,\cdots) \tag{5.85}$$

where δ_{rs} is the Kronecker delta. The natural modes satisfying Eqs.(5.85) are referred to as *normal modes*. It should be pointed out that normalization is not a unique process, and other definitions can be used. If the modes are normalized so that they satisfy Eq.(5.85), then upon integrating the left side of Eq.(5.81) and considering the boundary conditions leading to Eq.(5.84), it follows that

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx=\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx=\omega_r{}^2\delta_{rs} \qquad (r,s=1,2,\cdots) \tag{5.86}$$

Although Eqs.(5.80) and (5.84) were derived using the eigenvalue problem for a bar in bending, the same reasoning can be used to derive similar formulas for other types of vibratory systems, such as strings in transverse vibration.

We observe that the integrals (5.80) and (5.84) are symmetric in the indices r and s. This fact can be interpreted as being the counterpart for continuous systems of the fact that the matrices $[m]$ and $[k]$ are symmetric. Moreover, we observe that the integral (5.80) is always positive when $r=s$. This is the counterpart for continuous systems of the fact that the matrix $[m]$ is always positive definite. On the other hand, the integral (5.84) can be positive for $r=s$ or it can be zero without $Y_r(x)$ being identically zero. The counterpart for this is that the matrix $[k]$ can be positive definite or it can be positive semidefinite. Indeed, integral (5.84) is zero if $Y_r(x)$ is constant or a linear function of x, where the two configurations are recognized as the translational and rotational rigid-body modes, respectively. If the integral (5.84) is always positive for $r=s$ and if it becomes zero only if $Y_r(x)$ is identically zero, then the system is positive definite. On the other hand, if the system admits rigid-body modes, so that the integral (5.84) is generally positive for $r=s$ but can be zero without $Y_r(x)$ being identically zero, then the system is positive semidefinite. As might be expected, we conclude from Eq.(5.86) that the natural frequencies associated with the rigid-body modes are zero. Clearly, if both rigid-body modes are possible, then the zero natural frequency has multiplicity two. In this case the linear function of x, representing the rotational rigid-body mode, can be so chosen that the translational and rotational modes are orthogonal to one another. Moreover, by analogy with the approach for discrete systems, the conservation of linear momentum and angular momentum can be invoked to demonstrate that the rigid-body modes are orthogonal to the elastic modes. Hence, all the modes are orthogonal to one another, regardless whether they are rigid-body or elastic modes. Of course, in all cases this is not ordinary orthogonality but orthogonality with respect to the mass density.

As for discrete systems, an expansion theorem exists for continuous systems, where the theorem is based on the orthogonality property. The *expansion theorem* can be stated in the form: *Any function* $Y(x)$*, satisfying the boundary conditions of the problem and such that* $\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]$ *is a continuous function, can be represented by the absolutely and uniformly convergent series of the system eigenfunctions*

$$Y(x) = \sum_{i=1}^{\infty} c_r Y_r(x) \tag{5.87}$$

where the constant coefficients c_r are given by

$$c_r = \int_0^L m(x)Y(x)Y_r(x)dx \quad (r = 1, 2, \cdots) \tag{5.88}$$

If we recall that a periodic function can be represented by a Fourier series consisting of an infinite set of harmonic functions, the expansion theorem, Eqs.(5.87) and (5.88), can be regarded as a generalized Fourier series representation. In fact, in the special cases in which the eigenfunctions happen to be harmonic, the expansion theorem does reduce to a Fourier series representation.

Although we stated the expansion theorem in terms of the bending of a bar, the same theorem is applicable to an entire class of vibratory systems, including all the systems discussed in this chapter.

5.8 RESPONSE OF SYSTEMS BY MODAL ANALYSIS

The response of a system to initial excitation, external excitation, or both initial and external excitation can be obtained conveniently by modal analysis. The method is based on the expansion theorem of **Sec.5.7** and regards the response as a superposition of the system eigenfunctions multiplied by corresponding time-dependent generalized coordinates, in a manner entirely analogous to that for discrete systems. Of course, this necessitates first obtaining the solution of the system eigenvalue problem.

As an illustration of the method, let us consider a uniform bar in bending with both ends hinged (see **Fig.5.8**). The bar is subjected to the external distributed harmonic force $f(x,t) = F(x)\cos\omega t$ and the initial conditions

$$y(x,0) = y_0(x)\,,\quad \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = v_0(x) \tag{5.89}$$

From **Sec.5.5** we conclude that the boundary-value problem for a uniform bar reduces to

$$-EI\frac{\partial^4 y(x,t)}{\partial x^4} + F(x)\cos\omega t = m\frac{\partial^2 y(x,t)}{\partial t^2} \qquad 0 < x < L \tag{5.90}$$

where the flexural stiffness EI and mass per unit length m are constant. Because both ends are hinged, the boundary conditions are

$$y(0,t) = 0\,,\quad EI\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0} = 0 \tag{5.91}$$

$$y(L,t)=0\,,\quad EI\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=L}=0 \tag{5.92}$$

The eigenvalue problem associated with the system under consideration was solved in **Sec.5.6**. Hence, from **Sec.5.6**, we obtain the system natural frequencies

$$\omega_r=(r\pi)^2\sqrt{\frac{EI}{mL^4}}\qquad (r=1,2,\cdots) \tag{5.93}$$

and the natural modes

$$Y_r(x)=\sqrt{\frac{2}{mL}}\sin\frac{r\pi x}{L}\qquad (r=1,2,\cdots) \tag{5.94}$$

where the modes are clearly orthogonal. Note that the modes have been normalized so as to satisfy Eqs.(5.85) and (5.86), or

$$\int_0^L mY_r(x)Y_s(x)dx=\delta_{rs}\qquad (r,s=1,2,\cdots) \tag{5.95}$$

and

$$\int_0^L Y_s(x)EI\frac{d^4Y_r(x)}{dx^4}dx=\omega_r{}^2\delta_{rs}\qquad (r,s=1,2,\cdots) \tag{5.96}$$

According to modal analysis, we let the solution of Eq.(5.90) have the form

$$y(x,t)=\sum_{i=1}^{\infty}Y_r(x)q_r(t) \tag{5.97}$$

so that, inserting (5.97) into (5.90), we arrive at

$$\sum_{i=1}^{\infty}\ddot{q}_r(t)mY_r(x)+\sum_{i=1}^{\infty}q_r(t)EI\frac{d^4Y_r(x)}{dx^4}=F(x)\cos\omega t\qquad 0<x<L \tag{5.98}$$

Multiplying through by $Y_s(x)$ integrating over the domain, and considering Eqs.(5.95) and (5.96), we obtain the set of independent ordinary differential equations

$$\ddot{q}_r(t)+\omega_r{}^2q_r(t)=Q_r\cos\omega t\qquad (r=1,2,\cdots) \tag{5.99}$$

where

$$Q_r=\int_0^L F(x)Y_r(x)dx\qquad (r=1,2,\cdots) \tag{5.100}$$

are the generalized force associated with the generalized coordinates $q_r(t)$.

Equations (5.99) resemble the equation of motion of an undamped single-degree-of-freedom system subjected to external excitation. The response can be written in the general form

$$q_r(t)=\frac{Q_r}{\omega_r{}^2-\omega^2}\cos\omega t+A_r\cos\omega_r t+B_r\sin\omega_r t \tag{5.101}$$

Where A_r and B_r are the unknown parameters determined from the initial conditions as follows:

$$q_{r0}=q_r(0)\,,\quad \dot{q}_{r0}=\dot{q}_r(0) \tag{5.102}$$

The values of q_{r0} and $\dot{q}_{r0}$ can be obtained by using the initial conditions (5.89) in conjunction with Eq.(5.97), as follows:

$$y(x,0)=y_0(x)=\sum_{r=1}^{\infty}Y_r(x)q_r(0)=\sum_{r=1}^{\infty}Y_r(x)q_{r0} \tag{5.103}$$

Multiplying through by $mY_s(x)$, integrating over the domain $0<x<L$, and taking advantage of the orthogonality

conditions (5.95), we obtain

$$q_{r0} = \int_0^L m y_0(x) Y_r(x) dx \quad (r = 1, 2, \cdots) \tag{5.104}$$

Analogously, we conclude that

$$\dot{q}_{r0} = \int_0^L m v_0(x) Y_r(x) dx \quad (r = 1, 2, \cdots) \tag{5.105}$$

The initial generalized coordinates and velocities are calculated from Eq.(5.101)

$$q_r(0) = \frac{Q_r}{\omega_r^{\ 2} - \omega^2} + A_r \,, \quad \dot{q}_r(0) = \omega_r B_r \tag{5.106}$$

Then

$$A_r = q_{r0} - \frac{Q_r}{\omega_r^{\ 2} - \omega^2} \,, \quad B_r = \frac{\dot{q}_{r0}}{\omega_r} \tag{5.107}$$

The modal response $q_r(t)$ is obtained by inserting Eq.(5.107) into (5.101), with the result

$$q_r(t) = \frac{Q_r}{\omega_r^{\ 2} - \omega^2}(\cos\omega t - \cos\omega_r t) + q_{r0}\cos\omega_r t + \frac{\dot{q}_{r0}}{\omega_r}\sin\omega_r t \tag{5.108}$$

The general response is obtained from Eq.(5.97) as follows:

$$y(x,t) = \sum_{i=1}^{\infty} Y_r(x) \left\{ \frac{Q_r}{\omega_r^{\ 2} - \omega^2}(\cos\omega t - \cos\omega_r t) + q_{r0}\cos\omega_r t + \frac{\dot{q}_{r0}}{\omega_r}\sin\omega_r t \right\} \tag{5.109}$$

where $Y_r(x)$ is given by (5.94), ω_r by (5.93), $Q_r(t)$ by (5.100), q_{r0} by (5.104) and $\dot{q}_{r0}$ by (5.105). Of course, before evaluating Eq.(5.106), we must know the distributed forcing function $f(x,t)$ and the initial conditions $y(x,0) = y_0(x)$ and $\partial y(x,t)/\partial t|_{t=0} = v_0(x)$.