CHAPTER FOUR

MULTI-DEGREE-OF-FREEDOM SYSTEMS

4.1 INTRODUCTION

The systems with one and two degrees of freedom discussed in the first three chapters represented simple mathematical models of complex physical systems. These simple models were able to explain the dynamic behavior of the complex systems. Quite often, however, such idealizations are not possible, and mathematical models with a larger number of degrees of freedom must be considered.

Most vibrational systems encountered in physical situations have distributed properties, such as mass and stiffness. Systems of this type are said to possess an infinite number of degrees of freedom, because the system is fully described only when the motion is known at every point of the system. In many cases, the mass and stiffness distributions are highly nonuniform, and for such systems it may be more feasible to construct discrete mathematical models, which need only a finite number of parameters to describe the mass and stiffness properties. Moreover, a description of the motion of such discrete models requires only a finite number of coordinates. In this manner, systems with an infinite number of degrees of freedom are reduced to systems with only a finite number of degrees of freedom.

As shown in **Chap.3**, two-degree-of-freedom systems represent as significant departure from single-degree-of-freedom systems in the sense that the natural modes of vibration of the former have no counterpart in the latter. On the other hand, there is no basic difference between two- and multi-degree-of-freedom systems, except that the latter require more efficient treatment. Such treatment is made possible by the use of concepts of linear algebra in conjunction with matrix methods.

The motion of multi-degree-of-freedom systems is generally described by a finite set of simultaneous second-order ordinary differential equations. The solution of such sets of equations is not an easy task, even when the equations are linear and they possess constant coefficients, because the coupling terms require that the equations be solved simultaneously. Such a solution is not feasible, however, so that the most indicated approach is to remove the coupling by means of a coordinate transformation producing a set of independent second-order ordinary differential equations of motion. Then, the solution of the independent equations can be carried out individually by the methods of **Chap.2**. As demonstrated in **Chap.3**, the coordinate transformation decoupling the equations of motion is based on the modal vectors of the system, and the coordinates describing the independent equations are the natural coordinates. Finally, the solution of the simultaneous equations of motion is obtained by simply inserting the expressions for the natural coordinates into the equations describing the coordinate transformation in question. The process whereby the solution of a set of simultaneous equations of motion is carried out by transforming the simultaneous equations into a set of independent equations for the natural coordinates, solving the independent equations, and expressing the solution of the simultaneous equations as a linear combination of the modal vectors multiplied by the natural coordinates is known as modal analysis. In addition to permitting efficient solutions of otherwise difficult problems, modal analysis affords a great deal of insight into the behavior of complex vibrating systems. The emphasis in this chapter is placed on systematic ways of treating vibration problems associated with n-degree-of-freedom systems and on modern methods for obtaining numerical results by using high-speed electronic computers. The chapter generalizes and extends the material of **Chap.3**. It begins by deriving the differential equations of motion for an n-degree-of-freedom system. Concentrating on linear systems, the equations are conveniently expressed in matrix form. To reduce the system of simultaneous equations of motion to uncoupled form by means of a linear transformation, we must first obtain the modal matrix. This leads naturally to the eigenvalue problem and its solution, where the latter consists of the system natural frequencies and modal vectors. The understanding of the eigenvalue problem and the properties of its solution are greatly enhanced by the use of concepts from linear algebra. The responses of an n-degree-of-freedom system to initial excitation and externally applied forces are derived by modal analysis.

4.2 EQUATIONS OF MOTION FOR LINEAR SYSTEMS. MATRIX FORMULATION

We are interested in the motion of a multi-degree-of-freedom system in the neighborhood of an equilibrium position, where the equilibrium position is as defined in **Sec.1.4**. Without loss of generality, we assume that the equilibrium position is given by the trivial solution $q_1 = q_2 = \cdots = 0$. Moreover, we assume that the generalized displacements from the equilibrium position are sufficiently small that the force-displacement and force-velocity relations are linear, so that the generalized coordinates and their time derivatives appear in the differential equations of motion at most to the first power. This represents, in essence, the so called *small-motions assumption*, leading to a linear system of equations. In this section, we derive the differential equations of motion by applying Newton's second law.

Let us consider the linear system consisting of n masses m_i (i =1, 2, $\cdots$, n) connected by springs and dampers, as shown in **Fig.4.1a**, and draw the free-body diagram associated with the typical mass m_i (see **Fig.4.1b**). Because the motion takes place in one dimension, the total number of degrees of freedom of the system coincides with the number of masses n. In view of this, we can dispense with the vector notation, and denote the generalized coordinates representing the displacements of the masses m_i by $q_i(t)$ (i =1, 2, $\cdots$, n). Applying Newton's second law to a typical mass m_i we can write the differential equation of motion

$$m_i\ddot{q}_i(t) = -c_{i+1}\{\dot{q}_i(t)-\dot{q}_{i+1}(t)\} - c_i\{\dot{q}_i(t)-\dot{q}_{i-1}(t)\} - k_{i+1}\{q_i(t)-q_{i+1}(t)\} - k_i\{q_i(t)-q_{i-1}(t)\} + Q_i(t) \tag{4.1}$$

where $Q_i(t)$ presents the externally impressed force. Equation (4.1) can be rearranged in the form

$$m_i\ddot{q}_i(t) - c_{i+1}\dot{q}_{i+1}(t) + (c_i + c_{i+1})\dot{q}_i(t) - c_i\dot{q}_{i-1}(t) - k_{i+1}q_{i+1}(t) + (k_i + k_{i+1})q_i(t) - k_iq_{i-1}(t) = Q_i(t) \tag{4.2}$$

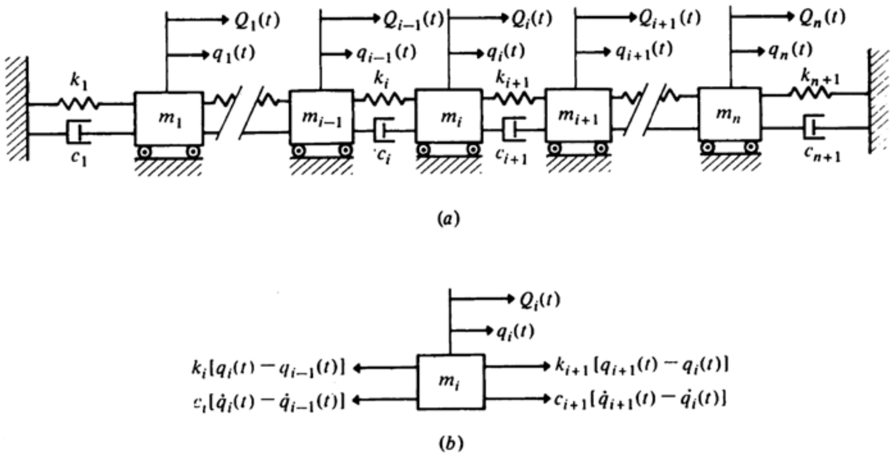

Figure 4.1

Next, let us introduce the notation

$$
\begin{aligned}
& m_{ij} = \delta_{ij} m_i \\
& c_{ij} = 0\,, \quad k_{ij} = 0 && j = 1, 2, \cdots, i-2, i+2, \cdots, n \\
& c_{ij} = -c_i\,, \quad k_{ij} = -k_i && j = i-1 \\
& c_{ij} = c_i + c_{i+1}\,, \quad k_{ij} = k_i + k_{i+1} && j = i \\
& c_{ij} = -c_{i+1}\,, \quad k_{ij} = -k_{i+1} && j = i+1
\end{aligned}
\tag{4.3}
$$

where m_{ij}, c_{ij} and k_{ij} are referred to as *mass*, *damping* and *stiffness coefficients*, respectively, and δ_{ij} is the *Kronecker delta*, defined as being equal to unity for $i = j$ and equal to zero for $i \neq j$. In view of notation (4.3), Eq.(4.2) can be used to express the complete set of equations of motion of the system as follows:

$$
\sum_{j=1}^{n} \left[m_{ij} \ddot{q}_j(t) + c_{ij} \dot{q}_j(t) + k_{ij} q_i(t) \right] = Q_i(t) \tag{4.4}
$$

which constitutes a set of n simultaneous second-order ordinary differential equations for the generalized coordinates $q_i(t)$ (i =1, 2, $\cdots$, n). We note that Eqs.(4.4) are quite general, and indeed they can accommodate other end conditions as well. For example, if the right end is free instead of fixed, then we can simply set $c_{n+1} = k_{n+1}$ =0 in Eqs.(4.3). Although at this particular point the notation (4.3) appears as an undesirable complication, its advantage lies in the fact that the use of double index for the coefficients permits writing Eqs.(4.4) in matrix notation. We shall have ample opportunity to work with the coefficients m_{ij}, c_{ij} and k_{ij} and to study their interesting and useful properties. In particular, it will be shown that the mass, damping and stiffness coefficients are symmetric

$$
m_{ij} = m_{ji}\,, \quad c_{ij} = c_{ji}\,, \quad k_{ij} = k_{ji} \qquad i\,, j = 1, 2, \ \cdots, \ n \tag{4.5}
$$

and that these coefficients control the system behavior, especially in the case of free vibration. Note that we encountered these coefficients for the first time in **Sec.3.2**.

In spite of the fact that Eqs.(4.4) possess constant coefficients, the general closed-form solution of the equations is extremely difficult to obtain, particularly because of the coupling introduced by the damping coefficients c_{ij}. Under special circumstances, however, the solution of Eqs.(4.4) is possible. In attempting a solution, it will prove convenient to write Eqs.(4.4) in matrix form. To this end, we arrange the coefficients m_{ij}, c_{ij} and k_{ij} in the following square matrices:

$$
\left[m_{ij} \right] = [m]\,, \quad \left[c_{ij} \right] = [c]\,, \quad \left[k_{ij} \right] = [k] \tag{4.6}
$$

and we note that the symmetry of the coefficients is expressed by the relations

$$
[m] = [m]^T\,, \quad [c] = [c]^T\,, \quad [k] = [k]^T \tag{4.7}
$$

where the superscript T denotes the transpose of the matrix in question. Moreover, we can arrange the generalized coordinates $q_i(t)$ and generalized impressed forces $Q_i(t)$ in the column matrices

$$
\{q_i(t)\} = \{q(t)\}\,, \quad \{Q_i(t)\} = \{Q(t)\} \tag{4.8}
$$

so that, using simple rules of matrix multiplication, Eqs.(4.4) can be written in the compact form

$$
[m]\{\ddot{q}(t)\} + [c]\{\dot{q}(t)\} + [k]\{q(t)\} = \{Q(t)\} \tag{4.9}
$$

As in **Sec.3.2**, the matrices $[m]$, $[c]$ and $[k]$ are called the *mass* or *inertia*, *damping* and *stiffness matrices*, respectively. The matrix $[m]$ is diagonal because of our particular choice of coordinates. For a different set of generalized coordinates $[m]$ is not necessarily diagonal.

The remainder of this chapter is devoted primarily to ways of obtaining the response of multi-degree-of-freedom systems.

Example 4.1

Consider the three-degree-of-freedom system of **Fig.4.2a** and derive the system differential equations of motion by using Newton's second law. The springs exhibit linear behavior and the dampers are viscous.

As shown in **Fig.4.2a**, the generalized coordinates $q_1(t)$, $q_2(t)$ and $q_3(t)$ represent the horizontal translations of masses m_1, m_2 and m_3, respectively, and $Q_1(t)$, $Q_2(t)$ and $Q_3(t)$ are the associated generalized externally applied forces. To derive the equations of motion by Newton's second law, we draw three free-body diagrams, associated with masses m_1, m_2 and m_3, respectively. They are all shown in **Fig.4.2b**, where the forces in the springs and dampers between masses m_1 and m_2 on the one hand and m_2 and m_3 on the other hand are the same in magnitude but opposite in direction. Application of Newton's second law for masses m_i (i =1, 2, 3) leads to the equations of motion

$$\begin{aligned}
&m_1\ddot{q}_1(t) = -c_2\{\dot{q}_1(t)-\dot{q}_2(t)\} - c_1\dot{q}_1(t) - k_2\{q_1(t)-q_2(t)\} - k_1q_1(t) + Q_1(t)\\
&m_2\ddot{q}_2(t) = -c_3\{\dot{q}_2(t)-\dot{q}_3(t)\} - c_2\{\dot{q}_2(t)-\dot{q}_1(t)\} - k_3\{q_2(t)-q_3(t)\} - k_2\{q_2(t)-q_1(t)\} + Q_2(t)\\
&m_3\ddot{q}_3(t) = -c_3\{\dot{q}_3(t)-\dot{q}_2(t)\} - k_3\{q_3(t)-q_2(t)\} + Q_3(t)
\end{aligned} \tag{a}$$

which can be rearranged in the form

$$\begin{aligned}
&m_1\ddot{q}_1(t) + (c_1+c_2)\dot{q}_1(t) - c_2\dot{q}_2(t) + (k_1+k_2)q_1(t) - k_2q_2(t) = Q_1(t)\\
&m_2\ddot{q}_2(t) - c_2\dot{q}_1(t) + (c_2+c_3)\dot{q}_2(t) - c_3\dot{q}_3(t) - k_2q_1(t) + (k_1+k_2)q_2(t) - k_3q_3(t) = Q_2(t)\\
&m_3\ddot{q}_3(t) - c_3\dot{q}_2(t) + c_3\dot{q}_3(t) - k_3q_2(t) + k_3q_3(t) = Q_3(t)
\end{aligned} \tag{b}$$

It is not difficult to see that Eqs.(b) can be expressed in the matrix form (4.9), where matrices $[m]$, $[c]$ and $[k]$ are given by

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{c}$$

$$[c] = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{d}$$

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{e}$$

which are clearly symmetric. Moreover, $[m]$ is diagonal.

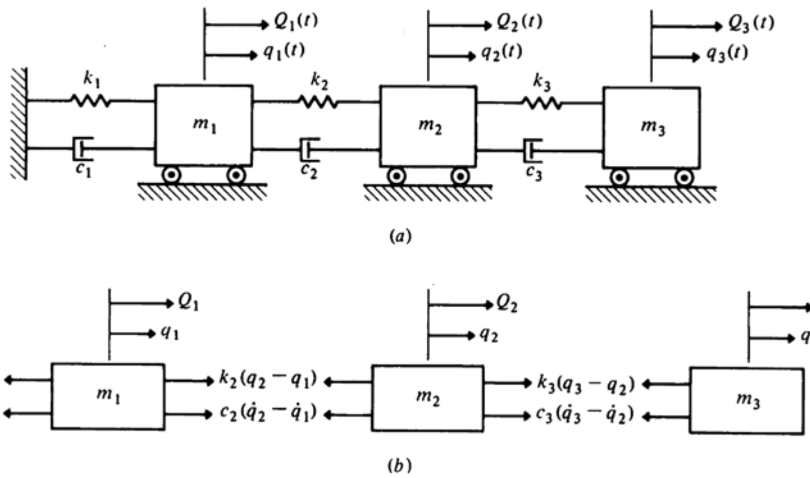

Figure 4.2

4.3 LINEAR TRANSFORMATIONS. COUPLING

Coupling depends on the coordinates used to describe the motion and is not a basic characteristic of the system. In this section, we discuss the ideas of coordinate transformations and coupling in broader terms.

Focusing our attention on the undamped case, we set $[c]=[0]$ in Eq. (4.9), where $[0]$ is the null square matrix of order n, and obtain the corresponding system of differential equations of motion

$$[m]\{\ddot{q}(t)\}+[k]\{q(t)\}=\{Q(t)\} \tag{4.10}$$

where $\{Q(t)\}$ is a column matrix whose elements are the n generalized externally impressed forces. For the purpose of this discussion, we consider the matrices $[m]$ and $[k]$ as arbitrary, except that they are symmetric and their elements constant. The column matrices $\{q\}$ and $\{Q\}$ represent n-dimensional vectors of generalized coordinates and forces, respectively.

It is clear from Eq.(4.10) that if $[m]$ is not diagonal, then the equations of motion are coupled through the inertial forces. On the other hand, if $[k]$ is not diagonal, the equations are coupled through the elastically restoring forces. In general Eq.(4.10) represents a set of n simultaneous linear second-order ordinary differential equations with constant coefficients. The solution of such a set of equations is not a simple task, and we wish to explore means of facilitating it. To this end, we express the equations of motion in a different set of generalized coordinates $\eta_j(t)$ ($j=1, 2, \cdots, n$) such that any coordinate $q_i(t)$ ($i=1, 2, \cdots, n$) is a linear combination of the coordinates $\eta_j(t)$. Hence, let us consider the linear transformation

$$\{q(t)\}=[u]\{\eta(t)\} \tag{4.11}$$

in which $[u]$ is a constant nonsingular square matrix, referred to as a *transformation matrix*. The matrix $[u]$ can be regarded as an operator transforming the vector $\{\eta\}$ into the vector $\{q\}$. Because $[u]$ is constant, we also have

$$\{\dot{q}(t)\}=[u]\{\dot{\eta}(t)\}, \quad \{\ddot{q}(t)\}=[u]\{\ddot{\eta}(t)\} \tag{4.12}$$

so that the same transformation matrix $[u]$ connects the velocity vectors $\{\dot{\eta}\}$ and $\{\dot{q}\}$ and the acceleration vectors $\{\ddot{\eta}\}$ and $\{\ddot{q}\}$. Inserting Eqs.(4.11) and (4.12) into (4.10), we arrive at

$$[m][u]\{\ddot{\eta}(t)\}+[k][u]\{\eta(t)\}=\{Q(t)\} \tag{4.13}$$

Next, we premultiply both sides of Eq. (4.13) by $[u]^T$ and obtain

$$[M]\{\ddot{\eta}(t)\}+[K]\{\eta(t)\}=\{N(t)\} \tag{4.14}$$

where the matrices

$$[M]=[u]^T[m][u]=[M]^T, \quad [K]=[u]^T[k][u]=[K]^T \tag{4.15}$$

are symmetric because $[m]$ and $[k]$ are symmetric. Moreover,

$$\{N(t)\}=[u]^T\{Q(t)\} \tag{4.16}$$

is an n-dimensional vector whose elements are the generalized forces N_i associated with the generalized coordinates η_i. Note that N_i are linear combinations of Q_j ($j=1, 2, \cdots, n$).

At this point we wish to return to the concept of coupling. If matrix $[M]$ is diagonal, then system (4.14) is said to be *inertially uncoupled*. On the other hand, if $[K]$ is diagonal, then the system is said to be *elastically uncoupled. The object of the transformation* (4.11) *is to produce diagonal matrices* $[M]$ *and* $[K]$ *simultaneously, because only then does the system consist of independent equations of motion*. Hence, if such a transformation matrix $[u]$ can be found, then Eq.(4.14) represents a set of n independent equations of the type

$$M_j\ddot{\eta}_j(t)+K_j\eta_j(t)=N_j(t) \quad (j=1, 2, \cdots, n) \tag{4.17}$$

where one of the two subscripts in M_{jj} and K_{jj} has been dropped because they are identical. Equations (4.17) have precisely the same structure as that of an undamped single-degree-of-freedom system [see Eq.(1.14) with $c=0$], and can be readily solved by the methods of **Chap.2**.

We state here (and prove later) that a linear transformation matrix $[u]$ diagonalizing $[m]$ and $[k]$ simultaneously does indeed exist. This particular matrix $[u]$ is known as the *modal matrix*, because it consists of the *modal vectors* or *characteristic vectors*, representing the *natural modes* of the system, and the coordinates $\eta_j(t)$ ($j=1, 2, \cdots, n$) are called *natural*, or *principal coordinates*. The procedure of solving the system of simultaneous differential equations of motion by transforming them into a set of independent equations by means of the modal matrix is generally referred to as *modal analysis*.

It is perhaps appropriate to pause at this point and reflect on the coordinate transformation (4.11), leading from equations of motion in terms of the coordinates $q_i(t)$ ($i=1, 2, \cdots, n$) to equations of motion in terms of the coordinates $\eta_j(t)$ ($j=1, 2, \cdots, n$). The new mass and stiffness matrices $[M]$ and $[K]$ are related to the original mass and stiffness matrices $[m]$ and $[k]$ by Eqs.(4.15). In the special case in which $[u]$ is the modal matrix, the matrices $[M]$ and $[K]$ become diagonal simultaneously and the matrix $[u]$ is said to be *orthogonal* (with respect to both $[m]$ and $[k]$). Moreover, in this case Eqs.(4.15) represent an *orthogonal transformation*, which is a special case of a similarity transformation and *the nature of the system does not change in similarity transformations*. But, because the new mass and stiffness matrices $[M]$ and $[K]$ are both diagonal, the equations of motion in terms of the coordinates $\eta_j(t)$ ($j=1, 2, \cdots, n$) become independent and very easy to solve. Hence, the linear transformation (4.11), in which $[u]$ is the modal matrix, permits an expeditious solution of the equations of motion.

It remains to find a way of determining the modal matrix $[u]$ for a given system. This can be accomplished by solving the eigenvalue problem associated with the matrices $[m]$ and $[k]$, a subject discussed in **Sec.4.4**. It should be pointed out that we already used a linear transformation of the type (4.11) to uncouple the equations of motion. Indeed, the vectors $\{u\}_1$ and $\{u\}_2$ multiplying the principal coordinates $q_1(t)$ and $q_2(t)$ in **Sec.3.4** were the modal vectors, and hence the columns of the modal matrix $[u]$. But, as pointed out in **Sec.3.3**, the modal vectors satisfy homogeneous algebraic equations, so that their magnitudes cannot be determined uniquely; only the ratios of the components of the modal vectors can. It is often convenient to choose the magnitude of the modal vectors so as to reduce the matrix $[M]$ to the identity matrix, which automatically reduces the matrix $[K]$ to the diagonal matrix of natural frequencies squared. This process is known as *normalization* and, under these circumstances, the modal matrix $[u]$ is said to be *orthonormal* (with respect to $[m]$ and $[k]$). In addition, the natural, or principal coordinates $\eta_j(t)$ ($j=1, 2, \cdots, n$) become *normal coordinates*.

Example 4.2

The modal matrix associated with the mass and stiffness matrices

$$[m]=m\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix},\quad [k]=k\begin{bmatrix}2&-1&0\\-1&3&-2\\0&-2&2\end{bmatrix} \tag{a}$$

can be shown to be

$$[u]=\frac{1}{\sqrt{m}}\begin{bmatrix}0.2691&-0.8782&0.3954\\0.5008&-0.2231&-0.8363\\0.5817&0.2992&0.2685\end{bmatrix} \tag{b}$$

Show that, when used as a transformation matrix, the matrix $[u]$ diagonalizes $[m]$ and $[k]$ simultaneously.

Inserting Eqs.(a) and (b) into Eqs.(4.15), we obtain the matrices

$$[M]=[u]^T[m][u]=\begin{bmatrix}0.2691 & -0.8782 & 0.3954\\ 0.5008 & -0.2231 & -0.8363\\ 0.5817 & 0.2992 & 0.2685\end{bmatrix}^T\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}0.2691 & -0.8782 & 0.3954\\ 0.5008 & -0.2231 & -0.8363\\ 0.5817 & 0.2992 & 0.2685\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \text{(c)}$$

and

$$[K]=[u]^T[k][u]=\frac{k}{m}\begin{bmatrix}0.2691 & -0.8782 & 0.3954\\ 0.5008 & -0.2231 & -0.8363\\ 0.5817 & 0.2992 & 0.2685\end{bmatrix}^T\begin{bmatrix}2 & -1 & 0\\ -1 & 3 & -2\\ 0 & -2 & 2\end{bmatrix}\begin{bmatrix}0.2691 & -0.8782 & 0.3954\\ 0.5008 & -0.2231 & -0.8363\\ 0.5817 & 0.2992 & 0.2685\end{bmatrix}$$
$$=\frac{k}{m}\begin{bmatrix}0.1392 & 0 & 0\\ 0 & 1.7458 & 0\\ 0 & 0 & 4.1152\end{bmatrix} \qquad \text{(d)}$$

which are clearly diagonal. Moreover, $[M]$ is the identity matrix, so that the modal matrix [u] is orthonormal. Consistent with this, the diagonal elements of $[K]$ are equal to the natural frequencies squared, as we shall verify later.

4.4 UNDAMPED FREE VIBRATION. EIGENVALUE PROBLEM

In **Sec.4.3**, we pointed out that, in the absence of damping, the equations of motion can be decoupled by using a transformation of coordinates, with the modal matrix acting as the transformation matrix. To determine the modal matrix, we must solve the so-called eigenvalue problem, a problem associated with free vibration, i.e., vibration in which the external forces are zero. In this section, we show how the free vibration problem leads directly to the eigenvalue problem, the solution of the latter yielding the natural modes of vibration. Then, we show that the natural motions, defined as motions in which the system vibrates in any one of the natural modes, can be identified as special cases of free vibration. Finally, we show that in the general case of free vibration, the motion can be regarded as a linear combination of the natural motions.

In the absence of external forces, $\{Q(t)\}=\{0\}$, Eq.(4.10) reduces to

$$[m]\{\ddot{q}(t)\}+[k]\{q(t)\}=\{0\} \qquad (4.18)$$

which represents a set of n simultaneous homogeneous differential equations of the type

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j(t)+\sum_{j=1}^{n} k_{ij}q_j(t)=0 \qquad (i=1,\ 2,\ \cdots,\ n) \qquad (4.19)$$

We are interested in a special type of solution of the set (4.19), namely, that in which all the coordinates $q_j(t)$ ($j=1,\ 2,\ \cdots,\ n$) execute synchronous motion. Physically, this implies a motion in which all the coordinates have the same time dependence, and the general configuration of the motion does not change, except for the amplitude, so that the ratio between any two coordinates $q_i(t)$ and $q_j(t)$, $i\neq j$, remains constant during the motion. Mathematically, this type of motion is expressed by

$$q_j(t)=u_j f(t) \qquad (4.20)$$

where u_j ($j=1,\ 2,\ \cdots,\ n$) are constant amplitudes and $f(t)$ is a function of time that is the same for all the coordinates $q_j(t)$. We are interested in the case in which the coordinates $q_j(t)$ represent stable oscillation, which implies that $f(t)$ must be bounded.

Inserting Eqs.(4.20) into (4.19), and recognizing that the function $f(t)$ does not depend on the index j, we obtain

$$\ddot{f}(t)\sum_{j=1}^{n} m_{ij}u_j + f(t)\sum_{j=1}^{n} k_{ij}u_j = 0 \qquad (i = 1, 2, \cdots, n) \tag{4.21}$$

Equations (4.21) can be written in the form

$$-\frac{\ddot{f}(t)}{f(t)} = \frac{\sum_{j=1}^{n} k_{ij}u_j}{\sum_{j=1}^{n} m_{ij}u_j} \qquad (i = 1, 2, \cdots, n) \tag{4.22}$$

with the implication that the time dependence and the positional dependence are separable, which is akin to the separation of variables for partial differential equations. Using the standard argument, we observe that the left side of (4.22) does not depend on the index i, whereas the right side does not depend on time, so that the two ratios must be equal to a constant. Assuming that $f(t)$ is a real function, the constant must be a real number. Denoting the constant by, λ, the set (4.22) yields

$$\ddot{f}(t) + \lambda f(t) = 0 \tag{4.23}$$

$$\sum_{j=1}^{n}\left(k_{ij} - \lambda m_{ij}\right)u_j = 0 \qquad (i = 1, 2, \cdots, n) \tag{4.24}$$

Let us consider a solution of Eq.(4.23) in the exponential form

$$f(t) = Ae^{st} \tag{4.25}$$

Introducing solution (4.25) into (4.23), we conclude that s must satisfy the equation

$$s^2 + \lambda = 0 \tag{4.26}$$

which has two roots

$$s_1, s_2 = \pm\sqrt{-\lambda} \tag{4.27}$$

If λ is a negative number (we have already concluded that it must be real), then s_1 and s_2 are real numbers, equal in magnitude but opposite in sign. In this case, Eq.(4.23) has two solutions, one decreasing and the other increasing exponentially with time. These solutions, however, are inconsistent with stable motion, so that the possibility that λ is negative must be discarded and the one that λ is positive considered. Letting $\lambda = \omega^2$, where ω is real, Eq.(4.27) yields

$$s_1, s_2 = \pm j\omega \tag{4.28}$$

so that the solution of Eq. (4.23) becomes

$$f(t) = A_1 e^{j\omega t} + A_2 e^{-j\omega t} \tag{4.29}$$

where A_1 and A_2 are generally complex numbers constant in value. Recognizing that $e^{j\omega t}$ and $e^{-j\omega t}$ represent complex vectors of unit magnitude, we conclude that solution (4.29) is harmonic with the frequency ω, and that it is the only acceptable solution of Eq.(4.23). This implies that if synchronous motion is possible, then the time dependent is harmonic. Because $f(t)$ is a real function, A_2 is the complex conjugate of A_1. It is easy to verify that solution (4.29) can be expressed in the form

$$\begin{aligned} f(t) &= A_1(\cos\omega t + j\sin\omega t) + A_2(\cos\omega t - j\sin\omega t) \\ &= (A_1 + A_2)\cos\omega t + j(A_1 - A_2)\sin\omega t = B_1\cos\omega t + B_2\sin\omega t \\ &= C\cos(\omega t - \phi) \end{aligned} \tag{4.30}$$

where B_1 and B_2, or C and ϕ are constant, ω is the frequency of the harmonic motion, and all three quantities being the same for every coordinate $q_j(t)$ ($j = 1, 2, \cdots, n$).

To complete the solution of Eqs.(4.19), we must determine the amplitudes u_j ($j=1, 2, \cdots, n$). To this end, we turn to Eqs.(4.24), which constitute a set of n homogeneous algebraic equations in the unknowns u_j, with $\lambda = \omega^2$ playing the role of a parameter. Not any arbitrary value of ω^2 permits a solution of Eqs.(4.24), but only a select set of n values. The problem of determining the values of ω^2 for which a nontrivial solution u_j ($j=1, 2, \cdots, n$) of Eqs.(4.24) exists is known as the *characteristic-value*, or *eigenvalue problem*.

It will prove convenient to write Eqs.(4.24) in the matrix form

$$[k]\{u\} = \omega^2 [m]\{u\} \tag{4.31}$$

Equation (4.31) represents the eigenvalue problem associated with matrices $[m]$ and $[k]$ and it possesses a nontrivial solution if and only if the determinant of the coefficients of u_j vanishes. This can be expressed in the form

$$\Delta\left(\omega^2\right) = \left|k_{ij} - \omega^2 m_{ij}\right| = 0 \tag{4.32}$$

where $\Delta\left(\omega^2\right)$ is called the *characteristic determinant*, with Eq.(4.32) itself being known as the *characteristic equation*, or *frequency equation*. It is an equation of degree n in ω^2, and it possesses in general n distinct roots, referred to as *characteristic values*, or *eigenvalues*. The n roots are denoted ω_1^2, ω_2^2, $\cdots$, ω_n^2 and the square roots of these quantities are the *system natural frequencies* ω_r ($r=1, 2, \cdots, n$). The natural frequencies can be arranged in order of increasing magnitude, namely, $\omega_1 \le \omega_2 \le \cdots \le \omega_n$. The lowest frequency ω_1 is referred to as the *fundamental frequency*, and for many practical problems it is the most important one. In general all frequencies ω_r are distinct and the equality sign never holds, except in *degenerate* cases (see discussion of such cases in **Sec.4.5**). It follows that there are n frequencies ω_r ($r=1, 2, \cdots, n$) in which harmonic motion of the type (4.30) is possible.

Associated with every one of the frequencies ω_r there is a certain nontrivial vector $\{u\}_r$ ($r=1, 2, \cdots, n$) whose elements u_{ir} are real numbers, where $\{u\}_r$ is a solution of the eigenvalue problem, such that

$$[k]\{u\}_r = \omega_r^{\,2} [m]\{u\}_r \tag{4.33}$$

The vectors $\{u\}_r$ ($r=1, 2, \cdots, n$) are known as *characteristic vector*, or *eigenvectors*. The eigenvectors are also referred to as *modal vectors* and represent physically the so-called *natural modes*. These vectors are unique only in the sense that the ratio between any two elements u_{ir} and u_{jr} is constant. The value of the elements themselves is arbitrary, however, because Eq. (4.31) is homogeneous, so that if $\{u\}_r$ is a solution of the equation, then $\alpha_r \{u\}_r$ is also a solution, where α_r is an arbitrary constant. Hence, we can say that *the shape of the natural modes is unique, but the amplitude is not*.

If one of the elements of the eigenvector $\{u\}_r$ is assigned a certain value, then the eigenvector is rendered unique in an absolute sense, because this automatically causes an adjustment in the values of the remaining $n-1$ elements by virtue of the fact that the ratio between any two elements is constant. The process of adjusting the elements of the natural modes to render their amplitude unique is called *normalization*, and the resulting vectors are referred to as *normal modes*. A very convenient normalization scheme consists of setting

$$\{u\}_r^{\,T} [m]\{u\}_r = 1 \quad (r=1, 2, \cdots, n) \tag{4.34}$$

which has the advantage that it yields

$$\{u\}_r^{\,T} [k]\{u\}_r = \omega_r^{\,2} \quad (r=1, 2, \cdots, n) \tag{4.35}$$

This can be easily shown by premultiplying both sides of (4.33) by $\{u\}_r^{\,T}$. Note that if this normalization scheme is used, then the elements of $\{u\}_r$ have units of $M^{-1/2}$, where M represents symbolically the units of the elements m_{ij} of the inertia matrix $[m]$. This, in turn, establishes the units of the constant C in Eq. (4.30), as can be concluded from Eqs. (4.20).

Another normalization scheme consists of setting the value of the largest element of the modal vector $\{u\}_r$ equal to 1, which may be convenient for plotting the modes. Clearly, *the normalization process is devoid of physical significance and should be regarded as a mere convenience*.

In view of Eqs. (4.20) and (4.30), we conclude that Eq. (4.18) has the solutions

$$\{q(t)\}_r = \{u\}_r f_r(t) \quad (r=1, 2, \cdots, n) \tag{4.36}$$

where

$$f_r(t) = C_r \cos(\omega_r t - \phi_r) \quad (r=1, 2, \cdots, n) \tag{4.37}$$

in which C_r and ϕ_r rare constants of integration representing amplitudes and phase angles, respectively. Hence, the free vibration problem admits special independent solutions in which the system vibrates in any one of the natural modes. These solutions are referred to as *natural motions*. Because for a linear system the general solution is the sum of the individual solutions, we can write the general solution of Eq. (4.18) as a linear combination of the natural motions, or

$$\{q(t)\} = \sum_{r=1}^{n} \{q(t)\}_r = \sum_{r=1}^{n} \{u\}_r f_r(t) = [u]\{f(t)\} \tag{4.38}$$

where

$$[u] = [\{u\}_1 \quad \{u\}_2 \quad \cdots \quad \{u\}_n] \tag{4.39}$$

is the *modal matrix* and $\{f(t)\}$ is a vector whose components $f_r(t)$ are given by Eqs.(4.37). The constants C_r and ϕ_r ($r=1, 2, \cdots, n$) entering into $\{f(t)\}$ depend on the initial conditions $\{q(0)\}$ and $\{\dot{q}(0)\}$. In **Sec.4.6**, we obtain solution (4.38), together with the evaluation of the constants of integration, by a more formal approach, namely, by modal analysis.

It should be pointed out that motion characteristics as described above are typical of positive definite systems, i.e., system for which the mass and stiffness matrices are real, symmetric, and positive definite.

Example 4.3

Derive the equations of motion for the two-degree-of freedom system shown in **Fig.4.3**, obtain the natural frequencies and natural modes and write the general solution to the free-vibration problem.

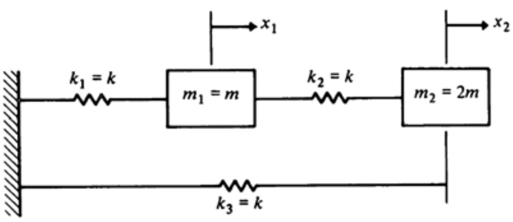

Figure 4.3

From **Fig.4.3**, we can write the equations of motion

$$\begin{aligned} m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2 x_2(t) = 0 \\ m_2\ddot{x}_2(t) - k_2 x_1(t) + (k_2 + k_3)x_2(t) = 0 \end{aligned} \tag{a}$$

so that the mass and stiffness matrices have the form

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}, \quad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \tag{b}$$

Introducing matrices (b) into Eq.(4.32), we arrive at the characteristic equation

$$\Delta(\omega^2) = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{vmatrix} = 2m^2\omega^4 - 6km\omega^2 + 3k^2 = 0 \tag{c}$$

Letting $k/m = \Omega^2$, Eq.(c) reduces to

$$\left(\frac{\omega}{\Omega}\right)^4 - 3\left(\frac{\omega}{\Omega}\right)^2 + \frac{3}{2} = 0 \tag{d}$$

which has the roots

$$\begin{matrix}\left(\dfrac{\omega_1}{\Omega}\right)^2 \\ \left(\dfrac{\omega_2}{\Omega}\right)^2\end{matrix} = \frac{3}{2} \mp \left\{\left(\frac{3}{2}\right)^2 - \frac{3}{2}\right\}^{1/2} = \frac{3}{2}\left(1 \mp \frac{1}{\sqrt{3}}\right) \tag{e}$$

so that the natural frequencies are

$$\begin{aligned} \omega_1 &= \left\{\frac{3}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right\}^{1/2}\Omega = 0.7962\sqrt{\frac{k}{m}} \\ \omega_2 &= \left\{\frac{3}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right\}^{1/2}\Omega = 1.5382\sqrt{\frac{k}{m}} \end{aligned} \tag{f}$$

To obtain the natural modes, we write Eq.(4.33) in the explicit form which in our case reduce to

$$\begin{aligned} (k_{11} - \omega_r^2 m_{11})u_{1r} + (k_{12} - \omega_r^2 m_{12})u_{2r} &= 0 \\ (k_{21} - \omega_r^2 m_{21})u_{1r} + (k_{22} - \omega_r^2 m_{22})u_{2r} &= 0 \end{aligned} \quad (r = 1, 2) \tag{g}$$

Which in our case reduce to

$$\begin{aligned} \left\{2 - \left(\frac{\omega_r}{\Omega}\right)^2\right\}u_{1r} - u_{2r} &= 0 \\ -u_{1r} + 2\left\{1 - \left(\frac{\omega_r}{\Omega}\right)^2\right\}u_{2r} &= 0 \end{aligned} \quad (r = 1, 2) \tag{h}$$

Because the problem is homogeneous, we can only solve for one element of a given modal vector in terms of the other. To this end, it is sufficient to solve only one of Eqs.(h) for each value of r. Which equation is solved is immaterial, because both yield the same result. We choose to solve the first equation. Letting $r = 1$, and using the value of $(\omega_1/\Omega)^2$ from Eq.(e), we obtain

$$u_{21} = \left\{2 - \left(\frac{\omega_1}{\Omega}\right)^2\right\}u_{11} = \left\{2 - \frac{3}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right\}u_{11} = 1.3660u_{11} \tag{i}$$

so that the first mode can be written in the form

$$\{u\}_1 = \begin{Bmatrix} 1.0000 \\ 1.3660 \end{Bmatrix} \tag{j}$$

where we normalized the mode by setting u_{11} =1.0000. In a similar fashion, we have

$$u_{22} = \left\{2 - \left(\frac{\omega_2}{\Omega}\right)^2\right\}u_{12} = \left\{2 - \frac{3}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right\}u_{12} = -0.3360u_{12} \tag{k}$$

leading to the second mode

$$\{u\}_2 = \begin{Bmatrix} 1.0000 \\ -0.3660 \end{Bmatrix} \tag{l}$$

where we set u_{12} =1.0000. Note that the second mode has a sign change, so that at some point between masses m_1

and m_2 the displacement is zero. Such a point is called a node. The modes are plotted in **Fig.4.4**.

According to Eq.(4.73), the solution of the free-vibration problem associated with Eqs.(a) can be written in the form

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = C_1 \begin{Bmatrix} 1.0000 \\ 1.3660 \end{Bmatrix} \cos\left(0.7962\sqrt{\frac{k}{m}}t - \phi_1\right) + C_2 \begin{Bmatrix} 1.0000 \\ -0.3660 \end{Bmatrix} \cos\left(1.5382\sqrt{\frac{k}{m}}t - \phi_2\right) \tag{m}$$

where C_1, C_2, ϕ_1 and ϕ_2 are determined from the initial conditions $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$ and $\dot{x}_2(0)$, as shown in **Sec.4.6**.

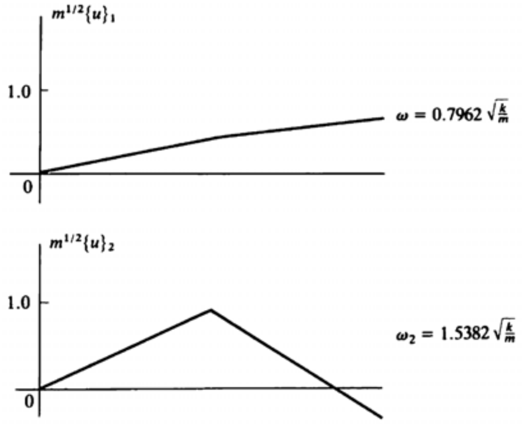

Figure 4.4

4.5 ORTHOGONALITY OF MODAL VECTORS. EXPANSION THEOREM

The natural modes possess a very important and useful property known as *orthogonality*. This is not an ordinary orthogonality, but an orthogonality with respect to the inertia matrix $[m]$ (and also with respect to the stiffness matrix $[k]$). Following is a proof of the orthogonality of the modal vectors $\{u\}_r$ (r =1, 2, $\cdots$, n).

Let us consider two distinct solutions $\omega_r^{\ 2}$, $\{u\}_r$, and $\omega_s^{\ 2}$, $\{u\}_s$ of the eigenvalue problem (4.31). These solutions can be written in the form

$$[k]\{u\}_r = \omega_r^{\ 2}[m]\{u\}_r \tag{4.40}$$

$$[k]\{u\}_s = \omega_s^{\ 2}[m]\{u\}_s \tag{4.41}$$

Premultiplying both sides of (4.40) by $\{u\}_s^{\ T}$ and both sides of (4.41) by $\{u\}_r^{\ T}$, we obtain

$$\{u\}_s^{\ T}[k]\{u\}_r = \omega_r^{\ 2}\{u\}_s^{\ T}[m]\{u\}_r \tag{4.42}$$

$$\{u\}_r^{\ T}[k]\{u\}_s = \omega_s^{\ 2}\{u\}_r^{\ T}[m]\{u\}_s \tag{4.43}$$

Next, let us transpose Eq.(4.43), here matrices $[m]$ and $[k]$ are symmetric, and subtract the result from (4.42) to obtain

$$\left(\omega_r^{\ 2} - \omega_s^{\ 2}\right)\{u\}_s^{\ T}[m]\{u\}_r = 0 \tag{4.44}$$

Because in general the natural frequencies are distinct, $\omega_r \neq \omega_s$, Eq.(4.44) is satisfied provided

$$\{u\}_s^{\ T}[m]\{u\}_r = 0 \qquad r \neq s \tag{4.45}$$

which is the statement of the *orthogonality condition* of the modal vectors. We note that the orthogonality is with respect to the inertia matrix $[m]$, which plays the role of a weighting matrix. Inserting Eq. (4.45) into (4.42), it is easy to see that the modal vectors are also orthogonal with respect to the stiffness matrix $[k]$,

$$\{u\}_s^{\ T}[k]\{u\}_r = 0 \qquad r \neq s \tag{4.46}$$

We stress again that the orthogonality relations (4.45) and (4.46) are valid only if $[m]$ and $[k]$ are symmetric. In many problems of practical interest the inertia matrix $[m]$ is diagonal, so that in these cases orthogonality condition

(4.45) is simpler to use. Regardless of whether $[m]$ is diagonal or not, however, condition (4.45) is ordinarily used rather than condition (4.46).

If the modes are normalized, then they are called *orthonormal*, and if the normalization scheme is according to Eq.(4.34), the modes satisfy the relation

$$\{u\}_r^T[m]\{u\}_s = \delta_{rs} \quad r, s = 1, 2, \cdots, n \tag{4.47}$$

where δ_{rs} is the Kronecker delta (see definition in **Sec.4.2**).

The question remains as to the case in which p natural frequencies are equal, where p is an integer such that $2 \le p \le n$. In this case, the modal vectors associated with the repeated eigenvalue are orthogonal to the remaining $n-p$ vectors, but in general they may not be orthogonal to one another. Fortunately, when the eigenvalue problem is defined in terms of two real symmetric matrices, such as the matrices $[m]$ and $[k]$ in the case at hand, the modal vectors corresponding to the repeated eigenvalue are orthogonal to one another. Indeed, according to a theorem of linear algebra, if an eigenvalue of a real symmetric matrix is repeated p times, then the matrix has p but not more than p mutually orthogonal eigenvectors corresponding to the repeated eigenvalue. The eigenvectors are not uniquely determined because, for repeated eigenvalues, any linear combination of the associated eigenvectors is also an eigenvector. In general, however, it is possible to choose p linear combinations of the eigenvectors corresponding to the repeated eigenvalue such that these combinations constitute mutually orthogonal eigenvectors, thus determining uniquely the eigenvectors in question. The above theorem is equally valid for the case in which the eigenvalue problem is defined in terms of two real symmetric matrices instead of one, if such a problem can be transformed into one in terms of a single real symmetric matrix by means of a linear transformation. The fact that the mass matrix $[m]$ is positive definite guarantees that a transformation to a single real symmetric matrix is always possible. Hence, *all the system eigenvectors are orthogonal*, regardless of whether the system possesses repeated eigenvalues or not. A system with repeated eigenvalues is referred to as *degenerate*.

The modal vectors can be conveniently arranged in a square matrix of order n, known as the *modal matrix* and having the form

$$[u] = [\{u\}_1 \quad \{u\}_2 \quad \cdots \quad \{u\}_n] \tag{4.48}$$

where $[u]$ is in fact the transformation matrix introduced in **Sec.4.3**. In view of definition (4.48), all n solutions of the eigenvalue problem, Eq.(4.33), can be written in the compact matrix equation

$$[k][u] = [m][u]\left[\omega_r^2\right] \tag{4.49}$$

where $\left[\omega_r^2\right]$ is a diagonal matrix of the natural frequencies squared. The fact that the modal matrix $[u]$ can be used as the transformation matrix uncoupling the system differential equations of motion is due to the orthogonality property of the natural modes. If the modes are normalized so as to satisfy Eqs.(4.47), then we can write

$$[u]^T[m][u] = [I], \quad [u]^T[k][u] = \left[\omega_r^2\right] \tag{4.50}$$

where $[I]$ is the unit matrix. Note that the second of Eqs.(4.50) follows directly from (4.49).

The eigenvectors $\{u\}_r$ ($r = 1, 2, \cdots, n$) form a *linearly independent set*, implying that any n-dimensional vector can be constructed as a linear combination of these eigenvectors. Physically this implies that any motion of the system can be regarded at any given time as a superposition of the natural modes multiplied by appropriate constants, where the constants are a measure of the degree of participation of each mode in the motion. The normal mode representation of the motion permits the transformation of a simultaneous set of differential equations of motion into an independent set, where the transformation matrix is the modal matrix $[u]$.

To prove that the set of vectors $\{u\}_r$ is linearly independent, we assume that the vectors are linearly dependent and arrive at a contradiction. For the vectors $\{u\}_r$ to be linearly dependent they must satisfy an equation of the type

$$c_1\{u\}_1 + c_2\{u\}_2 + \cdots + c_n\{u\}_n = \sum_{r=1}^{n} c_r\{u\}_r = \{0\} \tag{4.51}$$

where c_r (r =1, 2, $\cdots$, n) are nonzero constants. Premultiplying Eq. (4.51) by $\{u\}_s^T[m]$, we obtain

$$\sum_{r=1}^{n} c_r\{u\}_s^T[m]\{u\}_r = 0 \tag{4.52}$$

But the triple matrix product $\{u\}_s^T[m]\{u\}_r$ is equal to zero for $r \neq s$ and is different from zero for $r = s$. It follows that Eq.(4.52) can be satisfied only if $c_s = 0$. Repeating the operation n times, for s =1, 2, $\cdots$, n, we conclude that Eq.(4.52) can be satisfied only in the *trivial case* defined by $c_1 = c_2 = \cdots = c_n = 0$. Hence, the eigenvectors $\{u\}_r$ cannot satisfy any equation of the type (4.51), with the obvious conclusion that the system modal vectors are linearly independent.

Because the modal vectors $\{u\}_r$ cannot satisfy any equation of the type (4.51), we must have

$$\{u\} = c_1\{u\}_1 + c_2\{u\}_2 + \cdots + c_n\{u\}_n \neq 0 \tag{4.53}$$

where $\{u\}$ is called a *linear combination* of $\{u\}_1$, $\{u\}_2, \cdots, \{u\}_n$, with coefficients c_1, $c_2, \cdots, c_n$. The totality of linear combinations obtained by letting the coefficients c_1, $c_2, \cdots, c_n$ vary forms the *vector space* $\{u\}$, which is said to be *spanned* by $\{u\}_1$, $\{u\}_2, \cdots, \{u\}_n$. The set of vectors $\{u\}_r$ (r =1, 2, $\cdots$, n) is called a *generating system* of $\{u\}$ and, because the vectors are independent, the generating system is said to be a *basis* of $\{u\}$. Hence, any vector belonging to the space $\{u\}$ can be generated in the form of the linear combination (4.53). Physically this implies that *any possible motion of the system can be described as a linear combination of the modal vectors*. Considering Eq.(4.53) and the orthogonality condition in the form (4.47), the coefficients c_r can be obtained by writing simply

$$c_r = \{u\}_r^T[m]\{u\} \quad (r = 1, 2, \cdots, n) \tag{4.54}$$

where the coefficients c_r are a measure of the contribution of the associated modes $\{u\}_r$ to the motion $\{u\}$. Equations (4.53) and (4.54) are known in vibrations under the name of the *expansion theorem*. The derivation of the response of a system by modal analysis is based on the expansion theorem.

The natural frequencies ω_r and associated natural modes $\{u\}_r$ (r =1, 2, $\cdots$, n) are paired together and represent a unique characteristic of the system. Their values depend solely on the matrices $[m]$ and $[k]$. Every one of the pairs ω_r, $\{u\}_r$ can be excited independently of any other pair ω_s, $\{u\}_s$, $r \neq s$. For example, if the system is excited by a harmonic forcing function with frequency ω_r, then the system configuration will resemble the natural mode $\{u\}_r$. Of course, this represents a resonance condition, and the motion will tend to increase without bounds until the small-motions assumption is violated. On the other hand, if the system is imparted an initial excitation resembling the natural mode $\{u\}_r$ then the ensuing motion will be synchronous harmonic oscillation with the natural frequency ω_r. We shall devote ample time to the relation between the system response and the normal modes.

4.6 RESPONSE OF SYSTEMS TO INITIAL EXCITATION. MODAL ANALYSIS

Let us consider once again the free vibration of an undamped system. From **Sec.4.4**, we can write the equations of motion in the matrix form

$$[m]\{\ddot{q}(t)\} + [k]\{q(t)\} = \{0\} \tag{4.55}$$

where $\{q(t)\}$ is the vector of the generalized coordinates $q_i(t)$ (i =1, 2, $\cdots$, n). We seek now a formal solution of Eq.(4.55). A formal solution of Eq.(4.55) can be written in the form

$$\{q(t)\} = \eta_1(t)\{u\}_1 + \eta_2(t)\{u\}_2 + \cdots + \eta_n(t)\{u\}_n = \sum_{r=1}^{n} \eta_r(t)\{u\}_r = [u]\{\eta(t)\} \tag{4.56}$$

where $[u]$ is recognized as the modal matrix and $\{\eta(t)\}$ is the vector of the functions $\eta_r(t)$ (r=1, 2, $\cdots$, n). Equation (4.56) can be regarded as a linear transformation relating the vectors $\{q(t)\}$ and $\{\eta(t)\}$, where the transformation matrix $[u]$ is constant. It follows immediately from Eq. (4.57) that

$$\{\ddot{q}(t)\} = [u]\{\ddot{\eta}(t)\} \tag{4.57}$$

so that, inserting Eqs.(4.56) and (4.57) into Eq. (4.55), premultiplying the result by $[u]^T$, and considering Eqs.(4.50), we arrive at the independent set of equations

$$\ddot{\eta}_r(t) + \omega_r{}^2\eta_r(t) = 0 \quad (r=1, 2, \cdots, n) \tag{4.58}$$

where the variables $\eta_r(t)$ are identified as the *normal coordinates* of the system. By analogy with the free-vibration solution of an undamped single-degree-of freedom system, Eq.(1.37b), the solution of (4.58) is simply

$$\eta_r(t) = C_{1r}\cos\omega_r t + C_{2r}\sin\omega_r t \quad (r=1, 2, \cdots, n) \tag{4.59}$$

where C_{1r} and C_{2r} (r=1, 2, $\cdots$, n) are constants of integration. Inserting Eqs.(4.59) back into transformation (4.56), we obtain

$$\{q(t)\} = [u]\{\eta(t)\} = \sum_{r=1}^{n} \eta_r(t)\{u\}_r = \sum_{r=1}^{n} (C_{1r}\cos\omega_r t + C_{2r}\sin\omega_r t)\{u\}_r \tag{4.60}$$

so that the free vibration of a multi-degree-of-freedom system consists of a superposition of n modal vectors multiplied by harmonic functions with frequencies equal to the system natural frequencies and with amplitudes and phase angles depending on the initial conditions.

Letting $\{q(0)\}$ and $\{\dot{q}(0)\}$ be the initial displacement and velocity vectors, respectively, Eq.(4.60) leads to

$$\{q(0)\} = \sum_{r=1}^{n} C_{1r}\{u\}_r, \quad \{\dot{q}(0)\} = \sum_{r=1}^{n} C_{2r}\omega_r\{u\}_r \tag{4.61}$$

Premultiplying Eqs.(4.61) by $\{u\}_s{}^T[m]$, and considering the orthonormality relations, Eqs.(4.47), we can write

$$C_{1r} = \{u\}_r{}^T[m]\{q(0)\}, \quad C_{2r} = \frac{1}{\omega_r}\{u\}_r{}^T[m]\{\dot{q}(0)\} \tag{4.62}$$

so that, introducing Eqs.(4.62) into (4.60), we obtain the general expression

$$\{q(t)\} = \sum_{r=1}^{n}\left(\{u\}_r{}^T[m]\{q(0)\}\cos\omega_r t + \{u\}_r{}^T[m]\{\dot{q}(0)\}\frac{1}{\omega_r}\sin\omega_r t\right)\{u\}_r \tag{4.63}$$

which represents the response of the system to the initial displacement vector $\{q(0)\}$ and the initial velocity vector $\{\dot{q}(0)\}$.

Example 4.4

Consider the system of **Example 4.3** and verify that the natural modes are orthogonal. Then obtain the response to the initial conditions $\dot{x}_1(0) = v_0$, $x_1(0) = x_2(0) = \dot{x}_2(0) = 0$.

Inserting the modal vectors $\{u\}_1$ and $\{u\}_2$, Eqs.(j) and (l) of **Example 4.3**, into Eq.(4.47), we obtain

$$\{u\}_1{}^T[m]\{u\}_2 = \begin{Bmatrix} 1.0000 \\ 1.3660 \end{Bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1.0000 \\ -0.3660 \end{Bmatrix} = m(1.0000 - 2\times 1.3660 \times 0.3660) = 0 \tag{a}$$

so that the modes are verified as being orthogonal with respect to the mass matrix.

The general response of a multi-degree-of freedom system to initial excitation is given by Eq.(4.63). Of course, we must change the notation from $\{q(t)\}$, $\{q(0)\}$ and $\{\dot{q}(0)\}$ to $\{x(t)\}$, $\{x(0)\}$ and $\{\dot{x}(0)\}$, respectively. Because in our case $\{x(0)\}=\{0\}$, the response becomes

$$\{x(t)\}=\sum_{r=1}^{2}\left(\{u\}_r^T[m]\{\dot{x}(0)\}\frac{1}{\omega_r}\sin\omega_r t\right)\{u\}_r \tag{b}$$

where

$$\{\dot{x}(t)\}=\begin{Bmatrix}v_0\\0\end{Bmatrix} \tag{c}$$

Before using Eq.(b), however, we recall that the modal vectors must be normalized according to Eq.(4.34). Hence, let us assume that the normalized modal vectors have the form

$$\{u\}_1=\alpha_1\begin{Bmatrix}1.0000\\1.3660\end{Bmatrix},\quad \{u\}_2=\alpha_2\begin{Bmatrix}1.0000\\-0.3660\end{Bmatrix} \tag{d}$$

where the constants α_1 and α_2 are evaluated by using Eq.(4.34). Indeed, we can write

$$\begin{aligned}\{u\}_1^T[m]\{u\}_1&=\alpha_1^2\begin{Bmatrix}1.0000\\1.3660\end{Bmatrix}^T\begin{bmatrix}m&0\\0&2m\end{bmatrix}\begin{Bmatrix}1.0000\\1.3660\end{Bmatrix}=4.7320m\alpha_1^2=1\\ \{u\}_2^T[m]\{u\}_2&=\alpha_2^2\begin{Bmatrix}1.0000\\-0.3660\end{Bmatrix}^T\begin{bmatrix}m&0\\0&2m\end{bmatrix}\begin{Bmatrix}1.0000\\-0.3660\end{Bmatrix}=1.2679m\alpha_2^2=1\end{aligned} \tag{e}$$

yielding the constants

$$\alpha_1=\frac{0.4597}{\sqrt{m}},\quad \alpha_2=\frac{0.8881}{\sqrt{m}} \tag{f}$$

Hence, inserting the above values into (d), we obtain the normal modes

$$\{u\}_1=\frac{1}{\sqrt{m}}\begin{Bmatrix}0.4597\\0.6280\end{Bmatrix},\quad \{u\}_2=\frac{1}{\sqrt{m}}\begin{Bmatrix}0.8881\\-0.3251\end{Bmatrix} \tag{g}$$

Next, let us recall from **Example 4.3** that the system natural frequencies are

$$\omega_1=0.7962\sqrt{\frac{k}{m}},\quad \omega_2=1.5382\sqrt{\frac{k}{m}} \tag{h}$$

and form

$$\begin{aligned}\frac{1}{\omega_1}\{u\}_1^T[m]\{\dot{x}(0)\}&=\frac{1}{0.7962\sqrt{k/m}}\frac{1}{\sqrt{m}}\begin{Bmatrix}0.4597\\0.6280\end{Bmatrix}^T\begin{bmatrix}m&0\\0&2m\end{bmatrix}\begin{Bmatrix}v_0\\0\end{Bmatrix}=0.5774\frac{mv_0}{\sqrt{k}}\\ \frac{1}{\omega_2}\{u\}_2^T[m]\{\dot{x}(0)\}&=\frac{1}{1.5382\sqrt{k/m}}\frac{1}{\sqrt{m}}\begin{Bmatrix}0.8881\\-0.3251\end{Bmatrix}^T\begin{bmatrix}m&0\\0&2m\end{bmatrix}\begin{Bmatrix}v_0\\0\end{Bmatrix}=0.5774\frac{mv_0}{\sqrt{k}}\end{aligned} \tag{i}$$

so that, introducing Eqs.(g) through (i) into (b), we obtain the response

$$\begin{aligned}\{x(t)\}&=\left(0.5774\frac{mv_0}{\sqrt{k}}\sin 0.7962\sqrt{\frac{k}{m}}t\right)\frac{1}{\sqrt{m}}\begin{Bmatrix}0.4597\\0.6280\end{Bmatrix}+\left(0.5774\frac{mv_0}{\sqrt{k}}\sin 1.5382\sqrt{\frac{k}{m}}t\right)\frac{1}{\sqrt{m}}\begin{Bmatrix}0.8881\\-0.3251\end{Bmatrix}\\ &=v_0\sqrt{\frac{k}{m}}\begin{Bmatrix}0.2654\\0.3626\end{Bmatrix}\sin 0.7962\sqrt{\frac{k}{m}}t+v_0\sqrt{\frac{k}{m}}\begin{Bmatrix}0.5127\\-0.1877\end{Bmatrix}\sin 1.5382\sqrt{\frac{k}{m}}t\end{aligned} \tag{j}$$

and note that the elements of $\{x(t)\}$ have units of length, as should be expected.

As a matter of interest, let us calculate the velocity vector $\{\dot{x}(t)\}$. Differentiating Eq.(j) with respect to time, we have simply

$$\{\dot{x}(t)\}=v_0\begin{Bmatrix}0.2113\\0.2887\end{Bmatrix}\cos 0.7962\sqrt{\frac{k}{m}}t+v_0\begin{Bmatrix}0.7887\\-0.2887\end{Bmatrix}\cos 1.5382\sqrt{\frac{k}{m}}t \tag{k}$$

Letting $t=0$ in Eq.(k), we obtain $\{\dot{x}(0)\}=\{v_0 \quad 0\}^T$, thus verifying the validity of the solution.

4.7 RESPONSE OF SYSTEMS TO HARMONIC EXCITATION. MODAL ANALYSIS

Until now the discussion has been confined to the free vibration of discrete linear systems, placing the emphasis on the role of the natural modes in the construction of the system response. Indeed, in **Sec.4.6** we have shown how to determine the response of an undamped, n-degree-of-freedom system to initial excitation by means of modal analysis. However, modal analysis can be used to derive the response of undamped systems to any arbitrary excitation, whether in the form of initial excitation or externally impressed forces, and under certain circumstances also the response of viscously damped systems.

Considering first the response of an undamped system to harmonic excitation, we recall Eq.(4.10), representing the system differential equations of motion in the matrix form

$$[m]\{\ddot{q}(t)\}+[k]\{q(t)\}=\{F\}\cos\omega t \tag{4.64}$$

where $[m]$ and $[k]$ are $n\times n$ symmetric matrices, called correspondingly the inertia and stiffness matrix, and $\{q(t)\}$ and $\{F\}$ are the n-dimensional generalized coordinate and force vectors, respectively. Equation (4.64) constitutes a system of n simultaneous ordinary differential equations with constant coefficients. The equations are linear, and a solution can be obtained by the Laplace transformation method, at least in principle. In practice, however, the solution can be quite laborious, even for a two-degree-of-freedom system, so that a different method is advised. Indeed, a solution by modal analysis is appreciably less laborious. The basic idea behind modal analysis is to transform the simultaneous set of equations represented by (4.64) into an independent set of equations, where the transformation matrix is the modal matrix.

To obtain the steady-state solution of Eq.(4.64) by modal analysis, we must first solve the eigenvalue problem associated with matrices $[m]$ and $[k]$. The solution can be written in the general matrix form

$$[m][u]\left[\omega_r^{\,2}\right]=[k][u] \tag{4.65}$$

where $[u]$ is the modal matrix and $\left[\omega_r^{\,2}\right]$ the diagonal matrix of the natural frequencies squared. The modal matrix can be normalized so as to satisfy

$$[u]^T[m][u]=[I]\,,\quad [u]^T[k][u]=\left[\omega_r^{\,2}\right] \tag{4.66}$$

Next, we consider the linear transformation

$$\{q(t)\}=[u]\{\eta(t)\} \tag{4.67}$$

relating the vectors $\{q(t)\}$ and $\{\eta(t)\}$, where the vectors represent two different sets of generalized coordinates. Because $[u]$ is a constant matrix, a transformation similar to (4.67) exists between $\{\ddot{q}(t)\}$ and $\{\ddot{\eta}(t)\}$. Introducing transformation (4.67) into (4.64), premultiplying the result by $[u]^T$, and considering Eqs.(4.66), we obtain

$$\{\ddot{\eta}(t)\}+\left[\omega_r^{\,2}\right]\{\eta(t)\}=\{N\}\cos\omega t \tag{4.68}$$

where

$$\{N\}=[u]^T\{F\} \tag{4.69}$$

is an n-dimensional vector of generalized forces associated with the vector of generalized coordinates $\{\eta(t)\}$.

Equation (4.68) represents a set of n independent equations of the form

$$\ddot{\eta}_r(t)+\omega_r^{\,2}\eta_r(t)=N_r\cos\omega t \quad (r=1,2,\ \cdots,\ n) \tag{4.70}$$

where $\eta_r(t)$ are recognized as the system *normal coordinates*, introduced in **Sec.4.3**, and $N_r\cos\omega t$ are associated generalized forces. Equations (4.70) have the same structure as the differential equation of motion of a single-degree-of-freedom system of unit mass, natural frequency ω_r, and impressed force $N_r\cos\omega t$. Hence, the solution of Eqs.(4.70) can be obtained by the methods of **Chap.2**. Indeed, letting $\zeta \to 0$, $\omega_n \to \omega_r$ and $X_{st} \to N_r / \omega_r^2$ in Eq.(2.8), we can write the complete solution

$$\eta_r(t) = \frac{N_r}{\omega_r^2 - \omega^2} \cos \omega t \quad (r = 1, 2, \cdots, n) \tag{4.71}$$

And Eq.(4.67) can also be expressed in the form

$$\{q(t)\} = [u]\{\eta(t)\} = \sum_{r=1}^{n} \{u\}_r \eta_r(t) \tag{4.72}$$

where $\{u\}_r$ ($r = 1, 2, \cdots, n$) are the normalized modal vectors. Hence, the complete response of an undamped n-degree-of-freedom system can be obtained by inserting the normal coordinates (4.71) into Eq.(4.72). Note that the normal coordinates are sometimes called *modal coordinates*.

The response of a general viscously damped n-degree-of-freedom system represents a much more difficult problem. The difficulty can be traced to the coupling introduced by damping. To show this, we recall from **Sec.4.2** that the differential equations of motion of a viscously damped n-degree-of-freedom system can be written in the matrix form

$$[m]\{\ddot{q}(t)\} + [c]\{\dot{q}(t)\} + [k]\{q(t)\} = \{F\}\cos \omega t \tag{4.76}$$

where $[c]$ is the $n \times n$ symmetric damping matrix. The remaining quantities are as defined in Eq.(4.64). Using the transformation (4.67), Eq.(4.76) can be reduced to

$$\{\ddot{\eta}(t)\} + [C]\{\dot{\eta}(t)\} + [\omega_r^2]\{\eta(t)\} = \{N\}\cos \omega t \tag{4.77}$$

where

$$[C] = [u]^T [c][u] \tag{4.78}$$

is an $n \times n$ symmetric matrix, *generally nondiagonal*. Hence, in general the classical modal analysis does not lead to an independent system of differential equations of motion. Here, we shall consider some special cases in which $[C]$ is diagonal, or at least it can be treated approximately as diagonal.

In the special case in which $[c]$ is a linear combination of the matrices $[m]$ and $[k]$, namely, when

$$[c] = \alpha[m] + \beta[k] \tag{4.79}$$

where α and β are constants, matrix $[C]$ does indeed become diagonal,

$$[C] = \alpha[I] + \beta[\omega_r^2] \tag{4.80}$$

so that the set (4.77) reduces to an independent set of equations. The case described by Eq.(4.79) is known as *proportional damping*. Introducing the notation

$$[C] = [2\zeta_r \omega_r] \tag{4.81}$$

the n independent sets of equations can be written in the form

$$\ddot{\eta}_r(t) + 2\zeta_r \omega_r \dot{\eta}_r(t) + \omega_r^2 \eta_r(t) = N_r \cos \omega t \quad (r = 1, 2, \cdots, n) \tag{4.82}$$

where the notation has been chosen so as to render the structure of the equations identical to that of a viscously damped single-degree-of-freedom system of the type studied in **Chap.2**.

There are other special cases in which matrix $[C]$ becomes diagonal. They do not occur very frequently, however, and a discussion of these cases lies beyond the scope of this text.

A case occurring frequently is that in which damping is very small. In such a case, the coupling introduced by the off-diagonal terms of $[C]$ can be regarded as being a second-order effect, and a reasonable approximation can be obtained by discarding these off-diagonal terms. This amounts to regarding $[C]$ as diagonal, although in fact it is not.

When damping is not small, matrix $[C]$ is generally not diagonal, nor can it be regarded as diagonal.

Returning to Eqs.(4.82), we wish to obtain a solution by using the results of **Sec.2.2**. Letting $\omega_n \to \omega_r$ and

$X_{st} \to N_r / \omega_r^2$ in Eq.(2.8), and converting the notation to that used here, we can write simply

$$\eta_r(t) = \frac{N_r/\omega_r^2}{\left\{1-(\omega/\omega_r)^2\right\}^2 + \{2\zeta_r(\omega/\omega_r)\}^2}\left[\frac{2\zeta_r\omega}{\omega_r}\sin\omega t + \left\{1-\left(\frac{\omega}{\omega_r}\right)^2\right\}\cos\omega t\right] \quad (r=1,2,\ \cdots,\ n) \tag{4.83}$$

Hence, the solution of Eq.(4.76) is obtained by introducing Eqs.(4.83) into Eq. (4.72).

Example 4.5

Let the system shown in **Fig.4.3** be acted upon by the forces

$$F_1(t)=0,\quad F_2(t)=F_0\cos\omega t \tag{a}$$

derive the system response.

From **Example 4.3**, we can write the differential equations of motion

$$\begin{aligned} &m\ddot{x}_1(t)+2kx_1(t)-kx_2(t)=0 \\ &2m\ddot{x}_2(t)-kx_1(t)+2kx_2(t)=F_0\cos\omega t \end{aligned} \tag{b}$$

which can be expressed in the matrix form

$$[m]\{\ddot{x}(t)\}+[k]\{x(t)\}=\{f(t)\}=\{F\}\cos\omega t \tag{c}$$

where

$$[m]=m\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},\quad [k]=k\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \tag{d}$$

are the inertia and stiffness matrices for the system and

$$\{x(t)\}=\begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix},\quad \{f(t)\}=\{F\}\cos\omega t=\begin{Bmatrix}0\\ F_0\end{Bmatrix}\cos\omega t \tag{e}$$

are the two-dimensional displacement and force vectors, respectively.

To solve the problem by modal analysis, we must first solve the eigenvalue problem associated with $[m]$ and $[k]$. This was actually done in **Example 4.3**, from which we obtain the natural frequencies and natural modes

$$\begin{aligned} &\omega_1=0.7962\sqrt{\frac{k}{m}},\quad \{u\}_1=\frac{1}{\sqrt{m}}\begin{Bmatrix}0.4597\\ 0.6280\end{Bmatrix} \\ &\omega_2=1.5382\sqrt{\frac{k}{m}},\quad \{u\}_2=\frac{1}{\sqrt{m}}\begin{Bmatrix}0.8881\\ -0.3251\end{Bmatrix} \end{aligned} \tag{f}$$

where the modes were normalized in **Example 4.4** according to Eq.(4.34). The modal vectors can be arranged in the modal matrix

$$[u]=\frac{1}{\sqrt{m}}\begin{bmatrix}0.4597 & 0.8881\\ 0.6280 & -0.3251\end{bmatrix} \tag{g}$$

Following the procedure outlined earlier, we make use of the linear transformation

$$\{x(t)\}=[u]\{\eta(t)\} \tag{h}$$

where $\{\eta(t)\}$ is a two-dimensional vector of generalized coordinates, and obtain Eq.(4.68) in which $\{N\}$ is the two-dimensional vector of generalized forces having the form

$$\{N\}=[u]^T\{F\}=\frac{1}{\sqrt{m}}\begin{bmatrix}0.4597 & 0.8881\\ 0.6280 & -0.3251\end{bmatrix}^T\begin{Bmatrix}0\\ F_0\end{Bmatrix}=\frac{F_0}{\sqrt{m}}\begin{Bmatrix}0.6280\\ -0.3251\end{Bmatrix} \tag{i}$$

Inserting the elements of (i) into Eq.(4.71), we obtain

$$\eta_1(t) = 0.6280 \frac{F_0}{\sqrt{m}} \frac{1}{\omega_1^{\ 2} - \omega^2} \cos \omega t$$
$$\eta_2(t) = -0.3251 \frac{F_0}{\sqrt{m}} \frac{1}{\omega_2^{\ 2} - \omega^2} \cos \omega t \qquad \text{(j)}$$

Finally, introducing Eqs.(j) into (h), and considering Eqs.(f) and (g), we can write explicitly

$$\begin{aligned}
x_1(t) &= \frac{1}{\sqrt{m}} 0.4597 \eta_1(t) + \frac{1}{\sqrt{m}} 0.8881 \eta_2(t) \\
&= \frac{1}{\sqrt{m}} 0.4597 \times 0.6280 \frac{F_0}{\sqrt{m}} \frac{1}{\omega_1^{\ 2} - \omega^2} \cos \omega t + \frac{1}{\sqrt{m}} 0.8881 \times (-0.3251) \frac{F_0}{\sqrt{m}} \frac{1}{\omega_2^{\ 2} - \omega^2} \cos \omega t \\
&= \frac{1}{\sqrt{m}} 0.2887 \frac{F_0}{\sqrt{m}} \frac{1}{\omega_1^{\ 2} - \omega^2} \cos \omega t + \frac{1}{\sqrt{m}} 0.2887 \frac{F_0}{\sqrt{m}} \frac{1}{\omega_2^{\ 2} - \omega^2} \cos \omega t \\
x_2(t) &= \frac{1}{\sqrt{m}} 0.6280 \eta_1(t) + \frac{1}{\sqrt{m}} (-0.3251) \eta_2(t) \\
&= \frac{1}{\sqrt{m}} 0.6280 \times 0.6280 \frac{F_0}{\sqrt{m}} \frac{1}{\omega_1^{\ 2} - \omega^2} \cos \omega t + \frac{1}{\sqrt{m}} (-0.3251) \times (-0.3251) \frac{F_0}{\sqrt{m}} \frac{1}{\omega_2^{\ 2} - \omega^2} \cos \omega t \\
&= \frac{1}{\sqrt{m}} 0.3944 \frac{F_0}{\sqrt{m}} \frac{1}{\omega_1^{\ 2} - \omega^2} \cos \omega t + \frac{1}{\sqrt{m}} 0.1057 \frac{F_0}{\sqrt{m}} \frac{1}{\omega_2^{\ 2} - \omega^2} \cos \omega t
\end{aligned} \qquad \text{(k)}$$